



IRMA Lectures in Mathematics  
and Theoretical Physics 8

# AdS/CFT Correspondence: Einstein Metrics and Their Conformal Boundaries

Olivier Biquard  
Editor



European Mathematical Society

**IRMA Lectures in Mathematics and Theoretical Physics 8**

Edited by Vladimir G. Turaev

Institut de Recherche Mathématique Avancée  
Université Louis Pasteur et CNRS  
7 rue René Descartes  
67084 Strasbourg Cedex  
France

## **IRMA Lectures in Mathematics and Theoretical Physics**

Edited by Vladimir G. Turaev

This series is devoted to the publication of research monographs, lecture notes, and other materials arising from programs of the Institut de Recherche Mathématique Avancée (Strasbourg, France). The goal is to promote recent advances in mathematics and theoretical physics and to make them accessible to wide circles of mathematicians, physicists, and students of these disciplines.

Previously published in this series:

- 1 Deformation Quantization, *Gilles Halbout* (Ed.)
- 2 Locally Compact Quantum Groups and Groupoids, *Leonid Vainerman* (Ed.)
- 3 From Combinatorics to Dynamical Systems, *Frédéric Fauvet and Claude Mitschi* (Eds.)
- 4 Three courses on Partial Differential Equations, *Eric Sonnendrücker* (Ed.)
- 5 Infinite Dimensional Groups and Manifolds, *Tilman Wurzbacher* (Ed.)
- 6 *Athanase Papadopoulos*, Metric Spaces, Convexity and Nonpositive Curvature
- 7 Numerical Methods for Hyperbolic and Kinetic Problems, *Stéphane Cordier, Thierry Goudon, Michaël Gutnic and Eric Sonnendrücker* (Eds.)

Volumes 1–5 are available from Walter de Gruyter ([www.degruyter.de](http://www.degruyter.de))

# AdS/CFT Correspondence: Einstein Metrics and Their Conformal Boundaries

73rd Meeting of Theoretical Physicists and Mathematicians,  
Strasbourg, September 11–13, 2003

Olivier Biquard  
Editor



European Mathematical Society

Editor:

Olivier Biquard  
Institut de Recherche Mathématique Avancée  
Université Louis Pasteur et CNRS  
7 Rue René Descartes  
67084 Strasbourg Cedex  
France

2000 Mathematics Subject Classification 53Cxx, 81Txx, 53A30, 58J60, 83D05

ISBN 3-03719-013-2

Bibliographic information published by Die Deutsche Bibliothek  
Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data  
are available in the Internet at <http://dnb.ddb.de>.

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is  
concerned, specifically the rights of translation, reprinting, re-use of illustrations, recitation, broadcasting,  
reproduction on microfilms or in other ways, and storage in data banks. For any kind of use permission  
of the copyright owner must be obtained.

© 2005 European Mathematical Society

Contact address:

European Mathematical Society Publishing House  
Seminar for Applied Mathematics  
ETH-Zentrum FLI C4  
CH-8092 Zürich  
Switzerland

Phone: +41 (0)1 632 34 36  
Email: [info@ems-ph.org](mailto:info@ems-ph.org)  
Homepage: [www.ems-ph.org](http://www.ems-ph.org)

Typeset using the author's  $\text{\TeX}$  files: I. Zimmermann, Freiburg  
Printed in Germany

9 8 7 6 5 4 3 2 1

# Preface

In recent years the interaction between geometry and theoretical physics has become remarkable and had a deep influence on new ideas in these fields. One fairly recent meeting point between the two disciplines is the discovery in 1997 of the AdS/CFT (Anti de Sitter/Conformal Field Theory) correspondence by Maldacena, Witten, and others.

The mathematical side was initiated in 1985 by Fefferman and Graham: the aim was to find new conformal invariants by using a correspondence between (asymptotically hyperbolic) Einstein metrics and their conformal boundaries (the standard model being the real hyperbolic space with boundary the standard conformal structure on the sphere). On the physical side, the original statement provides a duality between the partition functions of a  $d$ -dimensional theory of quantum gravity and a  $(d - 1)$ -dimensional conformal field theory. As the reader probably already guessed, the mathematical object underlying quantum gravity is Einstein manifolds (possibly with additional fields), and CFTs live on conformal manifolds.

The field has now become extremely active both in mathematics and in physics since remarkable results and generalizations emerged from these ideas. That is why Vladimir Turaev and I organized in September 2003 the 73rd session of the traditional series of ‘Meetings between Physicists and Mathematicians’ on this subject. The meeting took place at the Institut de Recherche Mathématique Avancée (IRMA) in Strasbourg. The present volume contains solicited and refereed contributions based on this meeting.

Whilst it is almost impossible to include all the aspects of the AdS/CFT correspondence and its variants (e.g. dS/CFT), especially on the physical side, this volume endeavours to present at the same time survey papers giving a wide overview of the subject with results and questions (see the article of Anderson on the mathematical side, and the article of De Boer, Maoz and Naqvi on the physical side) and also more specialized papers.

The topics covered in the Riemannian case include conformal invariants (the obstruction tensor of Graham and Hirachi, related to the celebrated Fefferman–Graham asymptotic expansion), a new Hamiltonian method for holographic renormalization (Papadimitriou and Skenderis) and the mass (Herzlich). In the Lorentzian case, Solodukhin focuses on holographic description for Minkowski space, Anderson, Chruściel and Delay on static de Sitter Einstein solutions, and group theoretical methods for constructing Lorentzian AdS space-times are explored by Frances. Finally, Gauntlett, Martelli, Sparks and Waldram study a different aspect: supersymmetric AdS<sub>5</sub> solutions of M-theory, resulting in the construction of the first examples of irregular Sasaki–Einstein metrics.

Despite the progress of the recent years and the abundant literature, lots of fundamental mathematical and physical questions in the field remain open. It is my hope that this volume may serve as a bridge from mathematicians to physicists and from physicists to mathematicians, and also may focus the attention on this rich and fruitful area of mathematics and physics.

Finally, let me thank the whole staff of the IRMA for the organization of the conference in September 2003, especially Josiane Moreau and Claudine Orphanidès. It is also a pleasure to acknowledge financial support from IRMA and CNRS, and also from the European Research Training Network EDGE, via Gérard Besson in the Grenoble node.

Strasbourg, February 2005

*Olivier Biquard*

# Table of Contents

|                                                                                                                   |     |
|-------------------------------------------------------------------------------------------------------------------|-----|
| Preface .....                                                                                                     | v   |
| <i>Michael T. Anderson</i>                                                                                        |     |
| Geometric aspects of the AdS/CFT correspondence .....                                                             | 1   |
| <i>Jan de Boer, Liat Maoz and Asad Naqvi</i>                                                                      |     |
| Some aspects of the AdS/CFT correspondence .....                                                                  | 33  |
| <i>C. Robin Graham and Kengo Hirachi</i>                                                                          |     |
| The ambient obstruction tensor and $Q$ -curvature .....                                                           | 59  |
| <i>Ioannis Papadimitriou and Kostas Skenderis</i>                                                                 |     |
| AdS/CFT correspondence and geometry .....                                                                         | 73  |
| <i>Marc Herzlich</i>                                                                                              |     |
| Mass formulae for asymptotically hyperbolic manifolds .....                                                       | 103 |
| <i>Sergey N. Solodukhin</i>                                                                                       |     |
| Reconstructing Minkowski space-time .....                                                                         | 123 |
| <i>Michael T. Anderson, Piotr T. Chruściel and Erwann Delay</i>                                                   |     |
| Non-trivial, static, geodesically complete space-times with a negative cosmological constant II. $n \geq 5$ ..... | 165 |
| <i>Charles Frances</i>                                                                                            |     |
| The conformal boundary of anti-de Sitter space-times .....                                                        | 205 |
| <i>Jerome P. Gauntlett, Dario Martelli, James Sparks and Daniel Waldram</i>                                       |     |
| Supersymmetric AdS backgrounds in string and M-theory .....                                                       | 217 |





# Geometric aspects of the AdS/CFT correspondence

*Michael T. Anderson\**

*Department of Mathematics, S.U.N.Y. at Stony Brook  
Stony Brook, NY 11794-3651, U.S.A.  
email: [anderson@math.sunysb.edu](mailto:anderson@math.sunysb.edu)*

**Abstract.** We discuss classical gravitational aspects of the AdS/CFT correspondence, with the aim of obtaining a rigorous (mathematical) understanding of the semi-classical limit of the gravitational partition function. The paper surveys recent progress in the area, together with a selection of new results and open problems.

## Contents

|   |                                                                         |    |
|---|-------------------------------------------------------------------------|----|
| 0 | Introduction . . . . .                                                  | 1  |
| 1 | Conformally compact Einstein metrics . . . . .                          | 2  |
| 2 | Uniqueness issue . . . . .                                              | 7  |
| 3 | Existence issue . . . . .                                               | 12 |
| 4 | Role of $R \geq 0$ . . . . .                                            | 16 |
| 5 | Self-duality . . . . .                                                  | 19 |
| 6 | Continuation to de Sitter and self-similar vacuum space-times . . . . . | 21 |

## 0 Introduction

In this paper we discuss certain geometrical aspects of the AdS/CFT or Maldacena correspondence [42], [31], [47]. From a physics point of view, only the purely classical gravitational aspects of the correspondence on the AdS side are considered; thus no scalar,  $p$ -form or gauge fields, and no supergravity or string corrections are considered. On the CFT side, we only take account of the conformal structure on the boundary, and its corresponding stress-energy tensor. The discussion is also confined, by and large, to the Euclidean (or Riemannian) version of the correspondence. On the other hand,

---

\*Partially supported by NSF Grant DMS 0305865.

within this modest and restricted framework, there is quite a bit that is now known on a rigorous mathematical basis.

Broadly speaking, the AdS/CFT correspondence states the existence of a duality equivalence between gravitational theories (such as string or  $M$  theory) on anti-de Sitter spaces  $M$  and conformal field theories on the boundary at conformal infinity  $\partial M$ . In the restricted semi-classical framework above, the correspondence as formulated by Witten [47] states that

$$Z_{\text{CFT}}[\gamma] = \sum e^{-I(g)}, \quad (0.1)$$

where  $Z_{\text{CFT}}$  is the partition function of a CFT attached to a conformal structure  $[\gamma]$  on  $\partial M$ , and  $I(g)$  is the renormalized Einstein–Hilbert action of an Einstein metric  $g$  on  $M$  with conformal infinity  $[\gamma]$ . The sum is over all manifolds and metrics  $(M, g)$  with the given boundary data  $(\partial M, [\gamma])$ .

The main focus of the paper is on developing a framework in which the right side of (0.1) can be given a rigorous understanding. We survey existing work on geometrical aspects of the correspondence related to this issue, and discuss several new results, mostly in the later sections. Section 1 discusses general background information on the structure of conformally compact Einstein metrics, while §2 and §3 discuss uniqueness and existence issues for the Dirichlet–Einstein problem respectively. Section 4 explains the role of positive scalar curvature boundary data in the existence theory. In §5 it is shown that the correspondence becomes much more explicit in the case of self-dual or anti-self-dual metrics on 4-manifolds. Finally in §6 we discuss the continuation to de Sitter-type metrics, and the construction of globally self-similar solutions to the vacuum Einstein equations. Throughout the text, a number of open questions and problems are presented.

## 1 Conformally compact Einstein metrics

Let  $M$  be the interior of a compact  $n + 1$  dimensional manifold  $\bar{M}$  with non-empty boundary  $\partial M$ . A complete Riemannian metric  $g$  on  $M$  is  $C^{m,\alpha}$  conformally compact if there is a defining function  $\rho$  on  $\bar{M}$  such that the conformally equivalent metric

$$\tilde{g} = \rho^2 g \quad (1.1)$$

extends to a  $C^{m,\alpha}$  metric on the compactification  $\bar{M}$ . A defining function  $\rho$  is a smooth, non-negative function on  $\bar{M}$  with  $\rho^{-1}(0) = \partial M$  and  $d\rho \neq 0$  on  $\partial M$ . The induced metric  $\gamma = \tilde{g}|_{\partial M}$  is called the boundary metric associated to the compactification  $\tilde{g}$ . There are many possible defining functions, and hence many conformal compactifications of a given metric  $g$ , and so only the conformal class  $[\gamma]$  of  $\gamma$  on  $\partial M$ , called conformal infinity, is uniquely determined by  $(M, g)$ . Any manifold  $M$  carries many conformally compact metrics but we are mainly interested in Einstein

metrics  $g$ , normalized so that

$$\text{Ric}_g = -ng. \quad (1.2)$$

It is well known, and easily seen, that  $C^2$  conformally compact Einstein metrics are asymptotically hyperbolic (AH), in that  $|K_g + 1| = O(\rho^2)$ , where  $K_g$  denotes sectional curvature of  $(M, g)$ , and these two notions will be used interchangeably.

A compactification  $\bar{g} = \rho^2 g$  as in (1.1) is called geodesic if  $\rho(x) = \text{dist}_{\bar{g}}(x, \partial M)$ . These compactifications are especially useful for computational purposes, and for the remainder of the paper we work only with geodesic compactifications. Each choice of boundary metric  $\gamma \in [\gamma]$  determines a unique geodesic defining function  $\rho$  associated to  $(M, g)$ .

The Gauss Lemma gives the splitting

$$\bar{g} = d\rho^2 + g_\rho, \quad g = \rho^{-2}(d\rho^2 + g_\rho), \quad (1.3)$$

where  $g_\rho$  is a curve of metrics on  $\partial M$ . The Fefferman–Graham expansion [27] is a formal Taylor-type series expansion for the curve  $g_\rho$ . The exact form of the expansion depends on whether  $n$  is odd or even. For  $n$  odd, one has

$$g_\rho \sim g_{(0)} + \rho^2 g_{(2)} + \cdots + \rho^{n-1} g_{(n-1)} + \rho^n g_{(n)} + \rho^{n+1} g_{(n+1)} + \cdots \quad (1.4)$$

This expansion is even in powers of  $\rho$  up to order  $n - 1$ . The coefficients  $g_{(2k)}$ ,  $k \leq (n - 1)/2$  are locally determined by the boundary metric  $\gamma = g_{(0)}$ ; they are explicitly computable expressions in the curvature of  $\gamma$  and its covariant derivatives. The term  $g_{(n)}$  is transverse-traceless, i.e.

$$\text{tr}_\gamma g_{(n)} = 0, \quad \delta_\gamma g_{(n)} = 0, \quad (1.5)$$

but is otherwise undetermined by  $\gamma$ ; it depends on global aspects of the AH Einstein metric  $(M, g)$ . If  $n$  is even, one has

$$g_\rho \sim g_{(0)} + \rho^2 g_{(2)} + \cdots + \rho^{n-2} g_{(n-2)} + \rho^n g_{(n)} + \rho^n \log \rho h + \rho^{n+1} g_{(n+1)} + \cdots \quad (1.6)$$

Again the terms  $g_{(2k)}$  up to order  $n - 2$  are explicitly computable from the boundary metric  $\gamma$ , as is the transverse-traceless coefficient  $h$  of the first  $\log \rho$  term. The term  $h$  is an important term relating to the CFT on  $\partial M$ ; it is the metric variation of the conformal anomaly, cf. [22]. In odd dimensions, this always vanishes. The term  $g_{(n)}$  satisfies

$$\text{tr}_\gamma g_{(n)} = \tau, \quad \delta_\gamma g_{(n)} = \delta, \quad (1.7)$$

where  $\tau$  and  $\delta$  are explicitly determined by the boundary metric  $\gamma$  and its derivatives, but  $g_{(n)}$  is otherwise undetermined by  $\gamma$  and as above depends on the global geometry of  $(M, g)$ . In addition  $(\log \rho)^k$  terms appear at order  $> n$  when  $h \neq 0$ . Note also that these expansions depend on the choice of boundary metric  $\gamma \in [\gamma]$ . Transformation properties of the coefficients  $g_{(i)}$ ,  $i \leq n$ , and  $h$  as  $\gamma$  varies over  $[\gamma]$  are also readily computable, cf. [22].

Mathematically, these expansions are formal, obtained by compactifying the Einstein equations and taking iterated Lie derivatives of  $\bar{g}$  at  $\rho = 0$ ;

$$g_{(k)} = \frac{1}{k!} \mathcal{L}_T^{(k)} \bar{g}, \quad (1.8)$$

where  $T = \bar{\nabla} \rho$ . If  $\bar{g} \in C^{m,\alpha}(\bar{M})$ , then the expansions hold up to order  $m + \alpha$ . However, boundary regularity results are needed to ensure that if an AH Einstein metric  $g$  with boundary metric  $\gamma$  satisfies  $\gamma \in C^{m,\alpha}(\partial M)$ , then the compactification  $\bar{g} \in C^{m,\alpha}(\bar{M})$ .

In both cases  $n$  even or odd, the Einstein equations determine all higher order coefficients  $g_{(k)}$ ,  $k > n$ , in terms of  $g_{(0)}$  and  $g_{(n)}$ , so that an AH Einstein metric is formally determined by  $g_{(0)}$  and  $g_{(n)}$ . The term  $g_{(0)}$  corresponds to Dirichlet boundary data on  $\partial M$ , while  $g_{(n)}$  corresponds to Neumann boundary data (in analogy with the scalar Laplace operator). Thus, on AH Einstein metrics, the correspondence

$$g_{(0)} \rightarrow g_{(n)} \quad (1.9)$$

is analogous to the Dirichlet-to-Neumann map for harmonic functions. As discussed below, the term  $g_{(n)}$  corresponds essentially to the stress-energy tensor of the dual CFT on  $\partial M$ . However, the map (1.9) is only well defined per se if there is a unique AH Einstein metric with boundary data  $\gamma = g_{(0)}$ .

Understanding the correspondence (1.9) is one of the key issues in the AdS/CFT correspondence, restricted to the setting of the Introduction. Formally, knowing  $g_{(0)}$  and  $g_{(n)}$  allows one to construct the bulk gravitational field, that is the AH Einstein metric via the expansion (1.4) or (1.6). However, one needs to know that the expansions (1.4) or (1.6) actually converge to  $g_\rho$  for this to be of any use.

More significantly, if  $n$  is odd, given any real-analytic metric  $g_{(0)}$  and symmetric bilinear form  $g_{(n)}$  on  $\partial M$ , satisfying (1.5), there exists a unique  $C^\omega$  conformally compact Einstein metric  $g$  defined in a thickening  $\partial M \times [0, \varepsilon)$  of  $\partial M$ . In particular, the expansion (1.4) converges to  $g_\rho$ . A similar statement holds when  $n$  is even, cf. [6] for  $n = 3$  and [38] for general  $n$ . Thus, the terms  $g_{(0)}$  and  $g_{(n)}$  may be specified arbitrarily and independently of each other, subject only to the constraint (1.5) or (1.7), to give “local” AH Einstein metrics. This illustrates the global nature of the correspondence (1.9).

Next, we turn to the structure of the moduli space of AH Einstein metrics on a given  $(n + 1)$ -manifold  $M$ . Let  $E = E^\infty$  be the space of AH Einstein metrics on  $M$  which admit a  $C^\infty$  compactification  $\bar{g}$  as in (1.1). When  $n$  is even, we assume here that  $C^\infty$  means  $C^\infty$  polyhomogeneous, i.e.  $g_\rho = \phi(\rho, \rho^n \log \rho)$ , where  $\phi$  is a  $C^\infty$  function of the two variables. The topology on  $E$  is the  $C^\infty$  (polyhomogeneous) topology on metrics on  $\bar{M}$ . Let  $\mathcal{E} = E / \text{Diff}_1(\bar{M})$ , where  $\text{Diff}_1(\bar{M})$  is the group of  $C^\infty$  diffeomorphisms of  $\bar{M}$  inducing the identity on  $\partial M$ , acting on  $E$  in the usual way by pullback. (The CFT on  $\partial M$  is a gauge-type theory, and so is diffeomorphism co-

variant, not diffeomorphism invariant; hence, it is natural to require diffeomorphisms fixing  $\partial M$ ).

As boundary data, let  $\text{Met}(\partial M) = \text{Met}^\infty(\partial M)$  be the space of  $C^\infty$  metrics on  $\partial M$  and  $\mathcal{C} = \mathcal{C}(\partial M)$  the corresponding space of pointwise conformal classes. Occasionally we will also work with the spaces of real-analytic metrics  $C^\omega$ , or  $C^{m,\alpha}$ .

There is a natural boundary map,

$$\Pi: \mathcal{E} \rightarrow \mathcal{C}, \quad \Pi[g] = [\gamma], \quad (1.10)$$

which takes an AH Einstein metric  $g$  on  $M$  to its conformal infinity on  $\partial M$ .

One then has the following general result on the structure of  $\mathcal{E}$  and the map  $\Pi$ , building on previous work of Graham–Lee [30] and Biquard [13].

**Theorem 1.1** ([5], [6]). *Let  $M$  be a compact, oriented  $(n+1)$ -manifold with boundary  $\partial M$ . If  $\mathcal{E}$  is non-empty, then  $\mathcal{E}$  is a smooth infinite dimensional manifold. Further, the boundary map*

$$\Pi: \mathcal{E} \rightarrow \mathcal{C}$$

*is a  $C^\infty$  smooth Fredholm map of index 0. Thus the derivative  $D\Pi$  has finite dimensional kernel and cokernel, has closed range, and*

$$\dim \ker D\Pi = \dim \text{Coker } D\Pi.$$

Implicit in Theorem 1.1 is the boundary regularity statement that an AH Einstein metric with  $C^\infty$  conformal infinity has a  $C^\infty$  (polyhomogeneous) geodesic compactification. When  $n+1 = 4$ , this boundary regularity has been proved in [4], [6], including the cases of  $C^\omega$  and  $C^{m,\alpha}$  regularity. In dimensions  $n+1 > 4$ , boundary regularity has recently been proved by Chruściel et al. [19] in the  $C^\infty$  case. Moreover, when  $\gamma \in C^\omega$ , Kichenassamy [38] has proved that the expansions (1.4) and (1.6) converge to  $g_\rho$ .

In addition, the regular points of  $\Pi$ , that is the metrics in  $\mathcal{E}$  where  $D\Pi$  is an isomorphism, are open and dense in  $\mathcal{E}$ . Hence, if  $\mathcal{E} \neq \emptyset$ , then  $\Pi(\mathcal{E})$  has non-empty interior in  $\mathcal{C}$ . Thus, if  $M$  carries some AH Einstein metric, then it also carries a large set of them, parametrized at least by an open set in  $\mathcal{C}$ . The results above all hold with  $E$  in place of  $\mathcal{E}$ , without essential changes.

A basic issue in this area is the Dirichlet problem for AH Einstein metrics: given the topological data  $(M, \partial M)$ , and a conformal class  $[\gamma] \in \mathcal{C}$ , does there exist a unique AH Einstein metric  $g$  on  $M$ , with conformal infinity  $[\gamma]$ ? In terms of the boundary map  $\Pi$ , global existence is equivalent to the surjectivity of  $\Pi$ , while uniqueness is equivalent to the injectivity of  $\Pi$ .

For Riemannian metrics, the Einstein–Hilbert action is (usually) given by

$$I = -\frac{1}{16\pi G} \int_M (R - 2\Lambda) dv - \frac{1}{8\pi G} \int_{\partial M} H dA, \quad (1.11)$$

where  $R$  is the scalar curvature,  $\Lambda$  the cosmological constant and  $H$  is the mean curvature; (sometimes  $I$  is replaced by  $-I$ ). In the following, units are chosen so that  $16\pi G = 1$ .

Critical points of  $I$  satisfy the Einstein equations

$$\text{Ric} - \frac{R}{2}g + \Lambda g = 0, \quad (1.12)$$

and in the normalization (1.2),  $\Lambda = \frac{1}{2} \frac{n-1}{n+1} R = -\frac{1}{2}n(n-1)$ . However, both terms in (1.11) are infinite on metrics in  $\mathcal{E}$ . A number of schemes have been proposed by physicists to obtain a finite expression for  $I$  on  $\mathcal{E}$ . Among these, the most natural is the holographic renormalization, cf. [47], [34], [12], [22], described as follows. Given a fixed geodesic defining function  $\rho$  for  $g$ , let  $B(\varepsilon) = \{x \in (M, g) : \rho(x) \geq \varepsilon\}$ . If  $n$  is odd, from the expansion (1.4), one has an expansion of the volume of  $B(\varepsilon)$  in the form

$$\text{vol } B(\varepsilon) = v_{(0)}\varepsilon^{-n} + \cdots + v_{(n-1)}\varepsilon^{-1} + V + o(1), \quad (1.13)$$

where the terms  $v_{(k)}$  are explicitly computable from  $(\partial M, \gamma)$ . For an Einstein metric as in (1.2),  $R - 2\Lambda = -2n$ , so that

$$- \int_{B(\varepsilon)} (R - 2\Lambda) dv = 2n(v_{(0)}\varepsilon^{-n} + \cdots + v_{(n-1)}\varepsilon^{-1} + V) + o(1). \quad (1.14)$$

A similar expansion of the boundary integral in (1.11) over  $S(\varepsilon)$  has a form similar to (1.13), but with no constant term  $V$ . In fact local and covariant counterterms  $v_{(k)}(\varepsilon)$  for the integral in (1.14), and the corresponding boundary integral, can be constructed in terms of the metric  $\gamma_\varepsilon$  induced on the finite cut-off  $S(\varepsilon) = \partial B(\varepsilon)$ . These counterterms  $v_{(k)}(\varepsilon)$ , when suitably rescaled, converge to the counterterms  $v_{(k)}$  at infinity; this is one important aspect of the AdS/CFT correspondence.

Thus, define the renormalized action  $I^{\text{ren}}$  by

$$I^{\text{ren}}(g) = 2nV. \quad (1.15)$$

Similarly, if  $n$  is even, the expansion (1.6) gives

$$\text{vol } B(\varepsilon) = v_{(0)}\varepsilon^{-n} + \cdots + v_{(n-2)}\varepsilon^{-2} + L \log \varepsilon + V + o(1), \quad (1.16)$$

and again the terms  $v_{(k)}$  as well as  $L$  are explicitly computable from  $(\partial M, \gamma)$ . The coefficient  $L$ , equal to the integral of  $\text{tr } g_{(n)}$  over  $\partial M$ , agrees with the conformal anomaly of the dual CFT on  $\partial M$  in all known cases, [34]. The renormalized action is again defined by (1.15).

When  $n$  is odd,  $I^{\text{ren}}$  is independent of the choice of boundary metric  $\gamma \in [\gamma]$ , and thus  $I^{\text{ren}}$  is a smooth functional on the moduli space  $\mathcal{E}$ . When  $n$  is even, this is not the case;  $I^{\text{ren}}$  does depend on the choice of boundary metric, and so only gives a smooth functional on the space  $E$  (or  $E$  quotiented out by diffeomorphisms equal to the identity to order  $n$  at  $\partial M$ ). On the other hand,  $L$  is independent of the choice of boundary metric.

Consider the variation of  $I^{\text{ren}}$  at a given  $g \in E$ , i.e.

$$dI^{\text{ren}}(\dot{g}) = \frac{d}{dt} I^{\text{ren}}(g + t\dot{g}), \quad (1.17)$$

where  $\dot{g}$  is tangent to  $E$ . Thus,  $dI^{\text{ren}}$  may be considered as a 1-form on the manifold  $E$  (or  $\mathcal{E}$  when  $n$  is odd). In general, the differential  $dI^{\text{ren}}$  is the stress-energy (or energy-momentum) tensor of the action. Since Einstein metrics are critical points of  $I$  or  $I^{\text{ren}}$ , it is clear that  $dI^{\text{ren}}$  must be supported on  $\partial M$ . In [22] it is proved that

$$dI^{\text{ren}} = -\frac{n}{2}g_{(n)} + r_{(n)}, \quad (1.18)$$

where  $r_{(n)} = 0$  if  $n$  is odd, and is explicitly determined by  $\gamma = g_{(0)}$  if  $n$  is even. Thus

$$dI^{\text{ren}}(\dot{g}) = -\frac{n}{2} \int_{\partial M} \langle g_{(n)} + r_{(n)}, \dot{g}_{(0)} \rangle dv_{\gamma}, \quad (1.19)$$

where  $\dot{g}_{(0)}$  is the variation of the boundary metric  $\gamma$  induced by  $\dot{g}$ . On the gravitational side, the 1-form  $dI^{\text{ren}}$  is the (Brown–York) quasi-local stress-energy tensor  $T$ ; via the AdS/CFT correspondence, this corresponds to the expectation value  $\langle T \rangle$  of the stress-energy tensor of the dual CFT on  $\partial M$ .

In dimension 4, the renormalized action or volume can be given a quite different interpretation. Namely, by means of the Chern–Gauss–Bonnet theorem, one finds, on  $(M^4, g)$ ,

$$\int_M |W|^2 = 8\pi^2 \chi(M) - I^{\text{ren}}, \quad (1.20)$$

where  $\chi(M)$  is the Euler characteristic and  $W$  is the Weyl curvature of  $(M, g)$ , cf. [3]. Thus the renormalized action, which involves only the scalar curvature and volume, in fact controls much more; it controls  $L^2$  norm of the Weyl curvature  $W$  on-shell, i.e. on  $\mathcal{E}$ .

Since the left side of (1.20) is non-negative, an immediate consequence is that the renormalized action is uniformly bounded above on the full space  $\mathcal{E}$ , depending only on a lower bound for  $\chi(M)$ . Moreover,  $I^{\text{ren}}$  has an absolute maximum on hyperbolic metrics. Thus for example on the 4-ball  $B^4$ , the Poincaré metric has the largest action among all AH Einstein metrics on  $B^4$ . It is an interesting open question whether such a result also holds in higher dimensions.

## 2 Uniqueness issue

It is not unreasonable to believe that there should be some relation between the existence and uniqueness problems for the Einstein–Dirichlet problem. For example, the usual Fredholm alternative relates these two issues at the linearized level. In this section, we discuss the uniqueness question on the basis of a selection of examples.



**Example 2.1.** The first example of non-uniqueness was that found by Hawking–Page [32] in their analysis of the AdS–Schwarzschild, or AdS  $S^2$  black hole metric. In general dimensions, this is a curve of AH Einstein metrics on  $M = \mathbb{R}^2 \times S^{n-1}$  given by

$$g_m = V^{-1}dr^2 + Vd\theta^2 + r^2g_{S^{n-1}(1)}, \quad (2.1)$$

where

$$V(r) = 1 + r^2 - \frac{2m}{r^{n-2}}. \quad (2.2)$$

Here  $r \in [r_+, \infty)$ , where  $r_+$  is the largest root of  $V$ , and the circular parameter  $\theta \in [0, \beta]$ , where  $\beta = 4\pi r_+/(nr_+^2 + (n-2))$ . It is easy to see that the conformal infinity of  $g_m$  is given by the conformal class of the product metric on  $S^1(\beta) \times S^{n-1}(1)$ . The action of  $g_m$  is given by

$$I^{\text{ren}}(g_m) = -\beta\omega_{n-1}(r_+^n - r_+^{n-2} + c_n), \quad (2.3)$$

where  $c_n = 0$  if  $n$  is odd, and  $c_n = (-1)^{n/2} \frac{(n-1)!!^2}{n!}$  if  $n$  is even, with  $\omega_{n-1} = \text{vol } S^{n-1}(1)$ , cf. [25]. The stress-energy tensor of  $g_m$  is

$$dI^{\text{ren}} = -\left(r_+^n + r_+^{n-2} + \frac{2c_n}{n-1}\right) \text{diag}(1-n, 1, \dots, 1), \quad (2.4)$$

cf. [24]. As a function of  $m \in (0, \infty)$ , observe that  $\beta$  has a maximum value of  $\beta_0 = 2\pi/(n(n-2))^{1/2}$ , and for every  $m \neq m_0$ , there are two values  $m^\pm$  of  $m$  giving the same value of  $\beta$ . Thus two metrics have the same conformal infinity; the boundary map  $\Pi$  is a fold map (of the form  $x \rightarrow x^2$ ) along the curve  $g_m$ . Exactly the same formulas and behavior hold if  $S^{n-1}(1)$  is replaced by any closed Einstein manifold  $(N, g_N)$  with  $\text{Ric}_{g_N} = (n-2)g_N$ , with  $\omega_{n-1}$  replaced by  $\text{vol}_{g_N} N$ .

Note that if one allows the filling manifold  $M$  to change, a further metric has the same conformal infinity. Thus, choose  $M = B^{n+1}/\mathbb{Z} = S^1 \times \mathbb{R}^n$ , with the quotient of the hyperbolic metric  $g_{-1}$  on  $B^{n+1}$  by a translation isometry along a geodesic.

**Example 2.2.** As discussed in [4], a more drastic example of non-uniqueness occurs in the family of AdS toral black hole metrics. These are metrics on  $M = \mathbb{R}^2 \times T^{n-1}$ , where  $T^{n-1}$  is the  $(n-1)$ -torus, and the standard form of the metrics  $g_m$  is the same as in (2.1), with  $S^{n-1}(1)$  replaced by any flat metric on  $T^{n-1}$  and  $V$  in (2.2) replaced by  $V(r) = r^2 - \frac{2m}{r^{n-2}}$ ,  $\beta = 4\pi/nr_+$ . The conformal infinity of these metrics is the flat metric on the product  $S^1(\beta) \times T^{n-1}$ . Here  $\beta$  is monotone in  $m$ , and so on this space of metrics, the boundary map  $\Pi$  is 1-1. The action and stress-energy tensor are given by [24], [25]:

$$I^{\text{ren}}(g_m) = -\beta\omega_{n-1}r_+^n, \quad dI^{\text{ren}} = -r_+^n \text{diag}(1-n, 1, \dots, 1), \quad (2.5)$$

where  $\omega_{n-1} = \text{vol } T^{n-1}$ .

However, the actual situation is a little more subtle. The metrics  $g_m$  are all locally isometric, and so are isometric in the universal cover  $\mathbb{R}^2 \times \mathbb{R}^{n-1}$ ,

$$\tilde{g}_m = V^{-1}dr^2 + Vd\theta^2 + r^2g_{\mathbb{R}^{n-1}}. \quad (2.6)$$

Let  $(T^n, g_0)$  be any flat metric on the  $n$ -torus, and let  $\sigma$  be any simple closed geodesic in  $(T^n, \sigma)$ . Topologically, one may glue on a disc  $D^2 = \mathbb{R}^2$  onto  $\sigma$  to obtain a solid torus  $\mathbb{R}^2 \times T^{n-1}$ . Metrically, this is carried out as follows. Given  $\sigma$ , for any  $R$  sufficiently large, there exists  $S$ , and a covering map  $\pi = \pi_R: S^1 \times \mathbb{R}^{n-1} \rightarrow T^n$ , such that

$$\pi(S^1) = \sigma \quad \text{and} \quad \pi^*(S^2g_0) = V(R)d\theta^2 + R^2g_{\mathbb{R}^{n-1}}.$$

Here  $S$  is determined by the relation that  $V(R)^{1/2}\beta = SL(\sigma)$ , where  $L(\sigma)$  is the length of  $\sigma$  in  $(T^n, g_0)$ . Thus  $\pi$  takes the circle factor  $S^1$  to  $\sigma$  and maps the flat metric on  $S^1 \times \mathbb{R}^{n-1}$  to  $S^2g_0$  on  $T^n$ . The map  $\pi$  is given by dividing by a unique (twisted) isometric  $\mathbb{Z}^{n-1}$ -action on  $S^1 \times T^{n-1}$  and this action clearly extends to an isometric  $\mathbb{Z}^{n-1}$ -action on  $\tilde{g}_m$ . Letting  $R \rightarrow \infty$  and taking the corresponding limiting map  $\pi$  and  $\mathbb{Z}^{n-1}$ -action gives a “twisted” toral black hole metric

$$\hat{g}_m = V^{-1}dr^2 + [Vd\theta^2 + r^2g_{\mathbb{R}^{n-1}}]/\mathbb{Z}^{n-1} \quad (2.7)$$

on  $\mathbb{R}^2 \times T^{n-1}$  with conformal infinity  $(T^n, g_0)$ .

By varying the choices of  $\sigma$ , this gives infinitely many isometrically distinct AH Einstein metrics on  $\mathbb{R}^2 \times T^{n-1}$  with the same conformal infinity  $(T^n, \sigma)$ , so that  $\Pi$  is  $\infty$ -to-1; (note however that these metrics are all locally isometric). These metrics all lie in distinct components of the moduli space  $\mathcal{E}$ , so  $\mathcal{E}$  has infinitely many components on  $\mathbb{R}^2 \times T^{n-1}$ ; these components are permuted by the action of “large” diffeomorphisms on the boundary  $T^n$ , not isotopic to the identity, corresponding to the choices of simple closed geodesic  $\sigma$ .

One may take limits of any infinite sequence of these metrics, with fixed conformal infinity, by taking  $L(\sigma) \rightarrow \infty$ . All sequences have a unique limit given by the hyperbolic cusp metric

$$g_C = ds^2 + e^{2s}g_0, \quad (2.8)$$

on  $\mathbb{R} \times T^n$ . It is not difficult to compute exactly the action  $I^{\text{ren}}$  of  $\hat{g}_m$ , and it is easy to see that as  $L(\sigma) \rightarrow \infty$ ,

$$I^{\text{ren}}(\hat{g}_m) \rightarrow I^{\text{ren}}(g_C) = 0,$$

(corresponding to  $\beta \rightarrow \infty$ ).

**Remark 2.3.** Let  $(N, g_N)$  be any closed  $(n-1)$ -dimensional Einstein manifold, with  $\text{Ric}_{g_N} = -(n-1)g_N$ . Such metrics generate AdS black hole metrics just as in (2.1); thus

$$g_m = V^{-1}dr^2 + Vd\theta^2 + r^2g_N, \quad (2.9)$$

is an AH Einstein metric on  $\mathbb{R}^2 \times N$ , with conformal infinity  $S^1(\beta) \times (N, g_N)$ . Here  $V = -1 + r^2 - 2m/r^{n-2}$ ,  $r_+ > 0$  is the largest root of  $V$  and  $\beta = 4\pi r_+/(nr_+^2 - (n-2))$ . Again  $\beta$  is a monotone function of  $m$ . However,  $g_m$  is well defined for negative values of  $m$ ; in fact,  $g_m$  is well defined for  $m \in [m_-, \infty)$ , where

$$m_- = -\frac{1}{n-2} \left( \frac{n-2}{n} \right)^{n/2} \quad \text{with} \quad r_+ = \left( \frac{n-2}{n} \right)^{1/2}. \quad (2.10)$$

For the extremal value  $m_-$  of  $m$ ,  $V(r_+) = V'(r_+) = 0$ , and a simple calculation shows that the horizon  $\{r = r_+\}$  occurs at infinite (geodesic) distance to any given point in  $(M, g_{m_-})$ ; the horizon in this case is called degenerate (with zero surface gravity). Note that  $\beta(m_-) = \infty$ , so that the  $\theta$ -circles are in fact lines  $\mathbb{R}$ . As  $m$  decreases to  $m_-$ , the horizon diverges to infinity (in the opposite direction from the conformal infinity), while the length of the  $\theta$ -circles expands to  $\infty$ . Thus, the metric  $g_{m_-}$  is a complete metric on the manifold  $\mathbb{R} \times \mathbb{R} \times N = \mathbb{R}^2 \times N$ , but is no longer conformally compact; the conformal infinity is  $\mathbb{R} \times (N, g_N)$ . The action also diverges to  $-\infty$  as  $m \rightarrow m_-$ .

However, one may divide the infinite  $\theta$ -factor of the extremal metric  $g_{m_-}$  by  $\mathbb{Z}$  to obtain a complete metric  $\hat{g}$  on  $\mathbb{R} \times S^1 \times N$ . The metric  $\hat{g}$  is an AH Einstein metric with a single cusp-like end, and with conformal infinity  $S^1 \times (N, g_N)$ ; the length of the  $S^1$  factor may be arbitrary. This is a non-standard example of an AH Einstein metric, with a cusp-like end, as opposed to the standard hyperbolic cusp of (2.8).

Note however that in contrast to the situation in Example 2.2,  $\hat{g}$  does not arise as a limit of the curve  $g_m$  as  $m \rightarrow m_-$ ; as  $m \rightarrow m_-$ , the conformal infinity of  $g_m$  also degenerates.

**Example 2.4.** As a final example of non-uniqueness, let  $(N, g_0)$  be any complete, geometrically finite hyperbolic  $(n+1)$ -manifold. We assume that  $N$  has a conformal infinity  $(\partial N, \gamma_0)$ , as well as a finite number of parabolic or cusp ends, of the form (2.8). There are numerous examples of such manifolds in any dimension. If  $n$  is odd, the renormalized action and stress-energy tensor are given by

$$I^{\text{ren}}(N, g_0) = (-1)^m 2n \frac{2^{2m} \pi^m m!}{(2m)!} \chi(N), \quad dI^{\text{ren}}(N, g_0) = 0, \quad (2.11)$$

where  $n = 2m - 1$ , cf. [26]. If  $n$  is even, an explicit general formula for the renormalized action is not known, although of course it is finite, while  $dI^{\text{ren}}(N, g_0)$  is explicitly computable from the  $g_{(2)}$  term in (1.6), cf. [22] for example.

Now one may truncate and cap-off the cusp ends of  $(N, g)$  by glueing in solid tori  $\mathbb{R}^2 \times T^{n-1}$  with boundary  $\partial(\mathbb{R}^2 \times T^{n-1}) = T^n$ . In 3 dimensions, this is the process of hyperbolic Dehn filling, due to Thurston. Essentially exactly as in Example 2.2, a disc  $\mathbb{R}^2 = D^2$  can be attached to any simple closed geodesic  $\sigma$  in  $T^n$ .

Using the Dehn filling results in [7], G. Craig has recently shown [21] that all the cusp ends of  $N$  may be capped off in this way to produce infinitely many distinct manifolds  $M_i$ , with AH Einstein metrics  $g_i$ , all with conformal infinity given by

the original  $(\partial N, \gamma_0)$ . The construction implies that as the lengths of all geodesics  $\sigma_i$  diverge to infinity,  $(M_i, g_i)$  converges to the original manifold  $(N, g_0)$  in the Gromov–Hausdorff topology. Moreover, for all  $i$  large,  $I^{\text{ren}}(M_i, g_i) < I^{\text{ren}}(N, g_0)$ , and

$$I^{\text{ren}}(M_i, g_i) \rightarrow I^{\text{ren}}(N, g_0), \quad dI^{\text{ren}}(M_i, g_i) \rightarrow dI^{\text{ren}}(N, g_0), \quad (2.12)$$

in any dimension.

Taken together, the results above suggest that in general, there may be some difficulties in obtaining a well-defined (purely gravitational) semi-classical partition function  $Z_{\text{AdS}}$ . Analogous difficulties in defining the partition function for Euclidean quantum gravity have been discussed briefly in [9]. Thus, given  $(\partial M, \gamma)$ , the correspondence (0.1) requires summing over the moduli space of all AH Einstein manifolds  $(M, g)$  with the given boundary data  $(\partial M, \gamma)$ . In the infinite sets of AH Einstein metrics with fixed boundary data constructed above, the action  $I^{\text{ren}}$  is uniformly bounded above and converges to a limit. Moreover, all metrics are strictly stable, in the sense that the 2<sup>nd</sup> variation of the action among transverse-traceless perturbations vanishing at infinity is positive definite – there are no negative or zero eigenmodes present. Hence, all solutions contribute a definite positive amount to the partition function, and so the partition function is likely to be badly divergent. In slightly more detail, the zero-loop approximation to the partition function is badly divergent, while the one-loop approximation is also likely to be, unless there happen to be infinitely many other AH Einstein metrics giving rise to cancellations.

As will be seen below, this phenomenon does not occur when the boundary metric  $\gamma$  has positive scalar curvature, at least in dimension 4. This is perhaps then another reason for restricting the correspondence to boundary data of positive scalar curvature, as suggested by Witten [47], [48] for reasons related to the stability of the CFT.

However, a simple sum as in (0.1), even with the addition of higher loop corrections, may be ignoring certain important geometric information. Thus, suppose one had a finite dimensional connected moduli space  $\Lambda$  of solutions with fixed boundary data (so that there is a finite dimensional space of zero modes). In this case, one would not simply sum over the distinct solutions  $g_\lambda$ , but integrate the function  $e^{-I}$  over the (presumably finite volume) moduli space  $\Lambda$  with respect to the volume form induced by the  $L^2$  metric on the space of metrics. It seems reasonable and natural that a similar prescription should be used when one has an infinite sequence of isolated points  $\{g_i\}$  converging to a limit set  $X_\infty$ . The metrics  $g_i$  satisfy

$$\text{dist}_{L^2}(x_i, X_\infty) \rightarrow 0 \quad \text{as } i \rightarrow \infty,$$

and it is natural to include weight factors, depending on  $\text{dist}_{L^2}(x_i, X_\infty)$ , in the sum (0.1). What is not clear is exactly what weight factors one should choose.

Such infinite behavior in the gravitational partition function does appear in the remarkable paper of Dijkgraaf et al. [23], where the authors deal with the infinite family of BTZ black hole metrics, parametrized by relatively prime integers  $(c, d)$  corresponding to simple closed geodesics on  $T^2$ ; in the terminology above, these are

just the different hyperbolic Dehn fillings of a 2-torus. The partition function found in [23] does have suitable weight factors, leading to a convergent sum.

### 3 Existence issue

Next we turn to the global existence question, i.e. the surjectivity of the boundary map  $\Pi$ . Locally, the map  $\Pi$  is quite simple; its domain and target are smooth manifolds and the linearization  $D\Pi$  has finite equi-dimensional kernel and cokernel. However, globally the domain  $\mathcal{E}$  of  $\Pi$  is highly non-compact. To obtain a good global theory relating the domain and image of  $\Pi$ , one needs the map  $\Pi$  to be proper, i.e. for any compact set  $K \subset \mathcal{C}$ ,  $\Pi^{-1}(K)$  is compact in  $\mathcal{E}$ . In particular, for any  $[\gamma] \in \mathcal{C}$ ,  $\Pi^{-1}([\gamma])$  should be a compact set in  $\mathcal{E}$ . If this fails, for instance if  $\Pi^{-1}([\gamma])$  is not compact, one needs to understand the possible limit structures of metrics in  $\Pi^{-1}([\gamma])$ .

The lack of uniqueness or even finiteness discussed in §2 shows that in general  $\Pi$  is not proper. Any general results on the compactness of a space of Einstein metrics having a compact set of boundary metrics must rely on a simpler theory of compactness of Einstein metrics on closed manifolds, i.e. the study of moduli spaces of Einstein metrics on closed manifolds. In dimension 2, the moduli space of Einstein metrics is described by Teichmüller theory. Unfortunately, in general dimensions, such a theory does not exist, and seems out of current reach. However, there is quite a well-developed theory of moduli of Einstein metrics on closed manifolds in 4-dimensions, and this allows one to develop an analogous theory in the case of AH Einstein metrics.

The results described below are thus restricted to 4-manifolds  $M$ , with  $\partial M$  a 3-manifold. It seems reasonable that these results can be generalized to higher dimensions in the presence of extra symmetry via Kaluza–Klein type symmetry reductions, and progress in this direction would be very interesting.

Let  $\mathcal{C}^0$  be the space of conformal classes on a 3-manifold  $\partial M$  which have a *non-flat* representative metric of non-negative scalar curvature. (Of course not all 3-manifolds admit such a metric). Let  $\mathcal{E}^0 = \Pi^{-1}(\mathcal{C}^0)$ , and consider the restricted map

$$\Pi^0: \mathcal{E}^0 \rightarrow \mathcal{C}^0. \quad (3.1)$$

**Theorem 3.1** ([5]). *Let  $M$  be a 4-manifold satisfying*

$$H_2(\partial M) \rightarrow H_2(M) \rightarrow 0. \quad (3.2)$$

*Then the map  $\Pi^0$  in (3.1) is proper. Further,  $\Pi$  has a well-defined degree, given by*

$$\deg \Pi = \sum_{g_i \in \Pi^{-1}[\gamma]} (-1)^{\text{ind } g_i}. \quad (3.3)$$

Here  $[\gamma]$  is any regular value of  $\Pi$  in  $\mathcal{C}^0$  (recall the regular values are dense in  $\mathcal{C}^0$ ). Since  $\Pi^0$  is proper, the sum above is finite, and  $\text{ind}_{g_i}$  is the  $L^2$  index of  $g_i$ , that is the

dimension of the space of transverse-traceless  $L^2$  forms on which the 2<sup>nd</sup> variation of the action is negative definite (the number of negative eigenmodes).

It is obvious from the definition that

$$\deg \Pi \neq 0 \Rightarrow \Pi^0 \text{ is surjective.} \quad (3.4)$$

On the other hand,  $\Pi^0$  may or may not be surjective when  $\deg \Pi^0 = 0$ . Similarly,  $\deg \Pi^0 = \pm 1$  does not imply uniqueness of an AH Einstein metric with a given conformal infinity; it implies that generic boundary metrics have an odd number of AH Einstein filling metrics.

**Remark 3.2.** The condition (3.2) is used only to rule out degeneration of Einstein metrics to orbifolds. If one enlarges the space  $\mathcal{E}$  to include orbifold Einstein metrics  $\mathcal{E}_s$ , then  $\Pi^0$  is proper on the enlarged space  $\hat{\mathcal{E}} = \mathcal{E} \cup \mathcal{E}_s$ . However, it is not currently known if  $\hat{\mathcal{E}}$  has the structure of a smooth manifold, or of a manifold off a set of codimension  $k$ , for some  $k \geq 2$ , although one certainly expects this to be the case.

Theorem 3.1 implies in particular that there are only finitely many components  $C_\lambda$  of the moduli space  $\mathcal{E}^0$  for which  $\bigcap_\lambda \Pi(C_\lambda) \neq \emptyset$ , i.e. only finitely many components have a given boundary metric in common. Of course the degree  $\deg \Pi^0$  may depend on the choice of the component. This result also holds when one allows the manifold  $M$  to vary. Thus, given any  $K < \infty$ , there are only finitely many diffeomorphism types of 4-manifolds  $M$ , with a given boundary  $\partial M$ , with  $\chi(M) \leq K$ , and for which  $\Pi(\mathcal{E})$  contains any given element  $[\gamma] \in \mathcal{C}^0$ .

Thus, the infinities discussed in §2 cannot arise when  $R_\gamma > 0$ , and so the sum in (0.1) is essentially well defined. The sum could be infinite only if there exist  $(M_i, g_i)$  with  $\Pi(g_i) = \gamma$ , with  $\chi(M_i) \rightarrow +\infty$  (slightly analogous to the divergence of the string partition function). The role of the hypothesis  $R_\gamma > 0$  will be explained in more detail in §4, but we outline the general idea of the proof of Theorem 3.1 to explain how this condition arises.

By way of background, consider first the structure of the moduli space of unit volume Einstein metrics on a fixed closed 4-manifold. Modulo the possibility of orbifold degenerations, the overall structure of the moduli space is quite similar to that of the Teichmüller theory for the moduli space of constant curvature metrics on a closed surface. Recall that sequences  $\{g_i\}$  of unit area constant curvature metrics on a surface  $S$  have subsequences that either:

- Converge to a limit  $(S, g)$  (for example on  $S^2$  where the moduli space is a point),
- Collapse in the sense that the injectivity radius converges to 0 everywhere (as in the case of a divergent sequence of flat metrics on  $T^2$ ),
- Form cusps  $(N, g)$ ,  $N \subset S$ , as in the case of hyperbolic metrics on  $S$ . Thus  $S$  is a finite union of hyperbolic surfaces with cusp ends  $(N_k, g_k)$  together with a finite number of annuli  $\mathbb{R} \times S^1$  which are collapsed by the sequence  $\{g_i\}$ .

A similar basic trichotomy holds in dimension 4, cf. [2]. Thus, analogous to (1.20), the Chern–Gauss–Bonnet theorem implies a uniform upper bound on the  $L^2$  norm of

the Weyl curvature of an Einstein metric on a 4-manifold  $M$ . Using this, sequences of such metrics have subsequences that either converge, collapse, or form cusps as above, although one must allow also for the formation of orbifold singularities.

Now suppose instead that  $\{g_i\}$  is a sequence of AH Einstein metrics on a 4-manifold  $M$  for which the corresponding conformal infinities  $\gamma_i$  are contained in a compact set, so that

$$\gamma_i \rightarrow \gamma \quad (3.5)$$

(in a subsequence). Then using (3.5) and (1.20), one can again prove  $I^{\text{ren}}(g_i)$  remains uniformly bounded (so that, roughly speaking,  $I^{\text{ren}}$  is proper on  $\mathcal{C}$ ) and the trichotomy above still holds. The possibility of collapse can also be ruled out by the control on the boundary metrics. However, in general, the formation of cusps cannot be ruled out (as seen from the examples in §2).

More precisely, define an AH Einstein metric with cusps  $(N, g)$  to be a complete Einstein metric  $g$  on an  $(n + 1)$ -manifold  $N$  which has two types of ends, namely AH ends and cusp ends. A cusp end of  $(N, g)$  is an end  $E$  such that  $\text{vol}_g E < \infty$ , and hence is not conformally compact. The bound on the volume follows from the bound on the renormalized action, via (1.15). On any divergent sequence of points  $x_k \in E$ , the injectivity radius  $\text{inj}_g(x_k) \rightarrow 0$  as  $k \rightarrow \infty$ , so the metric  $g$  is collapsing at infinity in  $E$ . For instance, as discussed in Example 2.2, infinite sequences of twisted toral black hole metrics limit on a complete hyperbolic cusp metric (2.8). Similarly, infinite sequences in Example 2.4 limit on a complete hyperbolic manifold with cusp ends.

It will be seen in §4 that the hypothesis  $R_\gamma > 0$  (or  $R_\gamma \geq 0$  and  $\gamma$  not Ricci-flat) rules out the possible formation of cusps (in any dimension). This shows that  $\{g_i\}$  above has a subsequence converging to a limit AH Einstein metric  $g$  on  $M$ , so that  $\Pi^0$  is proper.

The examples of cusp formation discussed in §2 all take place on sequences of metrics  $g_i$  lying either in distinct components of  $\mathcal{E}$ , or on different smooth manifolds  $M_i$ . Another interesting open question is whether cusps can actually form within a given or fixed component of  $\mathcal{E}$ , on a fixed manifold  $M$ . On closed manifolds, it is clear that cusps can form at the endpoints of curves of Einstein metrics. For example, let  $M = \Sigma_{g_1} \times \Sigma_{g_2}$  be a product of surfaces of genus  $g_i \geq 2$ . Products of hyperbolic metrics on each surface are Einstein metrics on  $M$  and so as in Teichmüller theory there are smooth curves of Einstein metrics limiting on cusps  $(N, g)$  associated to  $M$ . However, it is not so easy to see, and in any case is unknown, if analogues of such constructions hold in the AH setting; compare also with Remark 2.3.

It would also be very interesting to know if the possible formation of cusps is restricted by the topology of the ambient manifold  $M$ . For example, the topological condition (3.2) rules out the formation of orbifold singularities. In the example above on  $M = \Sigma_{g_1} \times \Sigma_{g_2}$ , this is clearly the case; the fundamental group of the collapsed region is non-trivial and injects in the fundamental group of  $M$ . One might conjecture for instance that on the 4-ball  $B^4$ , or  $(n + 1)$ -ball  $B^{n+1}$ , cusp formation is not possible.

If one knows that no cusp formation is possible on limits of sequences in  $\mathcal{E} = \mathcal{E}(M)$  within a compact set of boundary metrics, then Theorem 3.1 holds in general, without the restriction to  $\mathcal{E}^0$ .

Consider briefly the situation in general where cusps may form. Given a fixed 4-manifold  $M$ , let  $\bar{\mathcal{E}}$  be the completion of  $\mathcal{E}$  in the Gromov–Hausdorff topology. Thus  $(X, g) \in \bar{\mathcal{E}}$  iff there is a sequence  $\{g_i\} \in \mathcal{E}$  such that  $(M, g_i) \rightarrow (X, g)$  in the (pointed) Gromov–Hausdorff topology. The analysis above implies that, in general,

$$\bar{\mathcal{E}} = \mathcal{E} \cup \mathcal{E}_s \cup \mathcal{E}_c, \quad (3.6)$$

where  $\mathcal{E}_s$  consists of orbifold AH Einstein metrics and  $\mathcal{E}_c$  consists of AH Einstein metrics with cusps associated to  $M$ . The boundary map  $\Pi$  extends to a continuous map

$$\bar{\Pi}: \bar{\mathcal{E}} \rightarrow \mathcal{C}, \quad (3.7)$$

and  $\bar{\Pi}$  is proper. However, the structure of  $\bar{\mathcal{E}}$  is not well understood. If for example  $\bar{\mathcal{E}}$  is a manifold off a singular set of codimension at least 2, then  $\deg \bar{\Pi}$  is well defined and is given by (3.3). In particular, (3.4) then holds. Moreover, one clearly has  $\deg \bar{\Pi} = \deg \Pi^0$ , when  $\mathcal{C}^0 \neq \emptyset$ . However, if  $\bar{\mathcal{E}}$  is something like a manifold with boundary, then  $\deg \bar{\Pi}$  is not well defined and the global behavior of  $\bar{\Pi}$  is less clear.

We now return to applications of Theorem 3.1 itself. The degree  $\deg \Pi^0$  is a smooth invariant of  $(M, \partial M)$ . In many specific cases, the degree can be calculated by means of the following isometry extension result; this result is very natural from the perspective of the AdS/CFT correspondence, and holds in all dimensions.

**Theorem 3.3** ([5]). *Let  $g$  be a  $C^\infty$  conformally compact Einstein metric on an  $(n+1)$ -manifold with conformal infinity  $[\gamma]$  on  $\partial M$ . Then any 1-parameter group of conformal isometries of  $(\partial M, \gamma)$  extends to a 1-parameter group of isometries of  $(M, g)$ .*

In particular, any AH Einstein metric whose conformal infinity is highly symmetric is itself highly symmetric. Einstein metrics which have a transitive or cohomogeneity 1 isometric group action have essentially been classified. Using this, the degree  $\deg \Pi^0$  can be computed in a number of interesting cases; the currently known results are summarized in the table on the next page.

Here  $(M, \partial M)$  is the given manifold with boundary and the existence of a seed metric implies  $\mathcal{E} \neq \emptyset$ . The degree is given for the component of  $\mathcal{E}$  containing the seed metric.

The manifold  $E_k$  is the  $\mathbb{R}^2$  bundle over  $S^2$  with Chern class  $k$ , while  $X_k$  is a resolution of the orbifold  $\mathbb{C}^2/\mathbb{Z}_k$  with  $c_1(X_k) < 0$ . The seed metric on  $X_k$  is an element of the family of self-dual AH Einstein metrics on  $X_k$  constructed by Calderbank–Singer [17].

Mazzeo–Pacard [43] have shown that if  $M_1$  and  $M_2$  admit an AH Einstein metric, then so does the boundary connected sum  $\hat{M} = M_1 \#_b M_2$ ; for  $\hat{M}$  one has  $\partial \hat{M} =$



| $M$                             | $\partial M$       | Seed Metric            | $\deg \Pi^0$ | $\Pi^0$ onto $\mathcal{C}^0$ |
|---------------------------------|--------------------|------------------------|--------------|------------------------------|
| $B^4$                           | $S^3$              | Poincaré               | 1            | Yes                          |
| $\mathbb{CP}^2 \setminus B^4$   | $S^3$              | AdS–Taub–Bolt          | 0            | No                           |
| $S^2 \times \mathbb{R}^2$       | $S^2 \times S^1$   | AdS–Schwarzschild      | 0            | No                           |
| $\mathbb{R}^3 \times S^1$       | $S^2 \times S^1$   | Poincaré/ $\mathbb{Z}$ | 1            | Yes                          |
| $E_k \rightarrow S^2, k \geq 2$ | $S^3/\mathbb{Z}_k$ | AdS–Taub–Bolt          | 1            | Yes                          |
| $X_k, k \geq 2$                 | $S^3/\mathbb{Z}_k$ | Self-dual CS metrics   | 0            | No                           |

$\partial M_1 \# \partial M_2$ . Further, if  $\mathcal{E}^0(M_i) \neq \emptyset$ , then  $\mathcal{E}^0(\hat{M}) \neq \emptyset$ . It would be interesting to determine the degree of  $\hat{M}$  in terms of the degree of each  $M_i$ .

**Remark 3.4.** (i) Let  $M$  be any  $(n+1)$ -manifold with  $\partial M = S^n$ . If  $M \neq B^{n+1}$ , then  $\Pi$  cannot be surjective onto  $\mathcal{C}^0$ ; in particular when  $n+1 = 4$ ,  $\deg \Pi^0 = 0$ . In fact the round metric  $g_{+1}$  on  $S^n$  cannot be in  $\text{Im } \Pi$ , for Theorem 3.3 implies that any such AH Einstein metric must be the hyperbolic metric on the ball. The same argument shows that the conformal class of the round product metric  $S^1(\beta) \times S^n(1)$ , for  $\beta > \beta_0 = 2\pi/(n(n-2))^{1/2}$ , is not in  $\text{Im } \Pi$  on any manifold  $M$  with  $\partial M = S^1 \times S^n$  except  $\mathbb{R}^n \times S^1$ , where it is uniquely realized by the hyperbolic metric.

In sum, one has the following examples of uniqueness results from Theorem 3.3. The Poincaré metric is the unique AH Einstein metric with boundary metric the round metric on  $S^n$ , while the AdS–Schwarzschild and (quotient) Poincaré metric are unique among AH Einstein metrics with boundary the round product on  $S^{n-1} \times S^1$ . Similarly, the only manifold carrying an AH Einstein metric with boundary metric the round metric on  $S^3/\mathbb{Z}_k$  is  $E_k$  when  $k \geq 2$ , and it is uniquely realized by the AdS–Taub–Bolt metric, cf. [32] for example.

(ii) By the same reasoning as above, any AH Einstein metric on  $M$  with conformal infinity a flat metric on the  $n$ -torus  $T^n$  is necessarily a twisted AdS toral black hole metric, as in Example 2.2. It follows from a simple Wick rotation argument that the Lorentzian AdS soliton metric of Horowitz–Myers [37], is the unique static AdS metric, when the conformal infinity is compactified to a flat torus; see also [29], [10] for previous work on the uniqueness of the AdS soliton.

## 4 Role of $R \geq 0$

In this section, we discuss the role of the hypothesis that the boundary metric  $\gamma$  on  $\partial M$  has non-negative scalar curvature. This condition was first suggested by Witten in [47], who pointed out that the corresponding CFT is unstable when  $R_\gamma < 0$ , suggesting that

the AdS/CFT correspondence may break down in this region. This result led shortly thereafter to the work of Witten–Yau [48], proving that  $\partial M$  is necessarily connected when  $\partial M$  carries a metric of positive scalar curvature, see also the work of Cai–Galloway [16] for a different proof which handles in addition the case of non-negative scalar curvature..

In this section we give a very elementary proof of the Witten–Yau result. In fact the result is stronger in that it gives a definite bound on the distance of any point in  $M$  to its boundary, in the geodesic compactification. This shows that the condition  $R_\gamma > 0$  not only proves that  $\partial M$  must be connected, but also prevents the formation of new, not necessarily conformally compact, boundary components in families of AH metrics.

Let  $g$  be an AH metric (not necessarily Einstein), on an  $(n+1)$ -manifold  $M$ , possibly with several boundary components; thus  $g$  is merely assumed to be  $C^3$  conformally compact. Given a fixed boundary component  $\partial_0 M$ , with associated boundary metric  $\gamma$ , let  $\rho$  be the geodesic defining function defined by  $(\partial_0 M, \gamma)$ , so that if  $\bar{g} = \rho^2 g$  is the associated (partial)  $C^2$  compactification of  $(M, g)$ , then  $\rho(x) = \text{dist}_{\bar{g}}(x, \partial_0 M)$ . Observe that if  $M$  has other boundary components, then these lie at infinite distance with respect to  $\bar{g}$  to any point in  $M$ . Note also that since  $g$  is  $C^2$  conformally compact,  $|\text{Ric}_g + ng| = O(\rho^2)$ .

**Theorem 4.1.** *Let  $\bar{g} = \rho^2 g$  be a partial geodesic compactification of an AH metric  $g$  on  $M$  satisfying*

$$\text{Ric}_g + ng \geq 0 \quad \text{and} \quad |\text{Ric}_g + ng| = o(\rho^2). \quad (4.1)$$

*If  $R_\gamma = \text{const} > 0$ , then for all  $x \in M$ ,*

$$\rho(x) \leq 4n(n-1)/R_\gamma. \quad (4.2)$$

*Proof.* Along the  $\bar{g}$ -geodesics normal to  $\partial_0 M$ , one has the Riccati equation

$$\bar{H}' + |\bar{K}|^2 + \overline{\text{Ric}}(T, T) = 0, \quad (4.3)$$

where  $T = \bar{\nabla}\rho$ ,  $\bar{K}$  is the 2nd fundamental form of the level sets  $S(\rho)$  of  $\rho$ , and  $\bar{H} = \text{tr } \bar{K}$  is the mean curvature, with  $\bar{H}' = \partial \bar{H} / \partial \rho$ . Thus  $\bar{K} = \bar{D}^2 \rho$ ,  $\bar{H} = \bar{\Delta} \rho$ . Here and in the following, the computations are with respect to  $\bar{g}$ . Standard formulas for conformal changes of metric give

$$\overline{\text{Ric}} = -(n-1) \frac{\bar{D}^2 \rho}{\rho} - \frac{\bar{\Delta} \rho}{\rho} \bar{g} + (\text{Ric}_g + ng) \geq -(n-1) \frac{\bar{D}^2 \rho}{\rho} - \frac{\bar{\Delta} \rho}{\rho} \bar{g}. \quad (4.4)$$

Hence

$$\bar{R} = -2n \frac{\bar{\Delta} \rho}{\rho} + (R + n(n+1))/\rho^2 \geq -2n \frac{\bar{\Delta} \rho}{\rho}. \quad (4.5)$$

In particular,  $\overline{\text{Ric}}(T, T) = -\frac{\bar{\Delta} \rho}{\rho} + (\text{Ric}_g + ng)(T, T)/\rho^2$ .

Dividing (4.3) by  $\rho$  then gives

$$\frac{(\bar{\Delta}\rho)'}{\rho} - \frac{\bar{\Delta}\rho}{\rho^2} + (\text{Ric}_g + ng)(T, T)/\rho^3 + \frac{|\bar{D}^2\rho|^2}{\rho} = 0,$$

so that

$$\left(\frac{\bar{\Delta}\rho}{\rho}\right)' = -\frac{|\bar{D}^2\rho|^2}{\rho} - (\text{Ric}_g + ng)(T, T)/\rho^3 \leq -\frac{|\bar{D}^2\rho|^2}{\rho}. \quad (4.6)$$

By Cauchy–Schwarz,  $|\bar{D}^2\rho|^2 \geq \frac{1}{n}(\bar{\Delta}\rho)^2$  and so, setting  $\phi = -\bar{\Delta}\rho/\rho$  one has

$$\phi' \geq \frac{1}{n}\rho\phi^2. \quad (4.7)$$

A simple computation using the Gauss equations at  $\partial M$  together with the fact that  $A = 0$  at  $\partial M$  implies that

$$(n-1)\phi(0) = \frac{1}{2}R_\gamma + \lim_{\rho \rightarrow 0} \frac{1}{\rho^2}[(\text{Ric}_g + ng)(T, T) - \frac{1}{2}(R + n(n+1))]. \quad (4.8)$$

The hypothesis (4.1) implies that the limit is 0, and hence  $\phi(0) > 0$ . A simple integration then gives

$$\rho^2 \leq 2n/\phi(0), \quad (4.9)$$

which gives (4.2).  $\square$

Note that if  $\gamma \in [\gamma]$  has non-negative scalar curvature, then there exists a representative  $\bar{\gamma} \in [\gamma]$  with constant non-negative scalar curvature, by the solution of the Yamabe problem.

As noted above, the estimate (4.2) immediately implies that  $\partial M$  is connected, since any other boundary component would have to lie at infinite  $\rho$ -distance to  $\partial_0 M$ . Similarly, simple topological arguments then imply

$$\pi_1(\partial M) \rightarrow \pi_1(M) \rightarrow 0.$$

More importantly for our purposes, the estimate (4.2) also immediately implies that AH Einstein metrics with cusps, as described in §3, cannot form as the limit of sequences of AH Einstein metrics with boundary metrics of uniformly positive scalar curvature. This explains then the role of  $\mathcal{C}^0$  in Theorem 3.1. A straightforward extension of the method above (based on the Cheeger–Gromoll splitting theorem) shows also that if  $R_\gamma = 0$  and  $\rho$  is unbounded on  $(M, g)$ , then  $(M, g)$  is isometric to

$$g = ds^2 + e^{2s}g_{N^n},$$

where  $g_{N^n}$  is Ricci-flat, cf. [16]. Thus when  $n = 3$ ,  $N$  must be flat so a finite cover of  $(M, g)$  is isometric to a hyperbolic cusp metric (2.8). In particular, this can only happen if  $(\partial M, \gamma)$  is flat.

**Remark 4.2.** Theorem 4.1 was proved in [4]. The proof is included here partly for completeness, and partly because the Lorentzian version of this result will be used in §6.

An elementary consequence of Theorem 4.1 is that a geometrically finite hyperbolic manifold with conformal infinity satisfying  $R_\gamma > 0$  has no parabolic ends.

## 5 Self-duality

The analysis in the previous sections describes the beginnings of a well-defined existence theory for the Einstein–Dirichlet problem, at least in 4-dimensions. From the point of view of the AdS/CFT correspondence, one would like however much more detailed information about the correspondence (1.9) of the Dirichlet and Neumann boundary data.

Again, restricting to dimension 4, a good deal more can be said, using the splitting of the curvature tensor into self-dual and anti-self-dual parts. Thus, let  $M = M^4$  be an oriented 4-manifold with boundary. As first noticed by Hitchin [35], given any  $C^2$  conformally compact metric  $g$  on  $M$  (not necessarily Einstein), a simple calculation using the Atiyah–Patodi–Singer index theorem gives

$$\frac{1}{12\pi^2} \int_M |W^+|^2 - |W^-|^2 = \sigma(M) - \eta(\gamma), \quad (5.1)$$

where  $W^\pm$  are the self-dual and anti-self-dual parts of the Weyl curvature,  $\sigma(M)$  is the signature of  $M$ , and  $\eta(\gamma)$  is the eta invariant of the conformal infinity  $(\partial M, \gamma)$ ; note that (5.1) is conformally invariant. Combining this with the formula (1.20) for the renormalized action of an AH Einstein metric gives the formula

$$I^{\text{ren}} \leq 8\pi^2 \chi(M) - 12\pi^2 |\sigma(M) - \eta(\gamma)|, \quad (5.2)$$

with equality if and only if  $g$  is self-dual (or anti-self-dual).

Hence if  $(M, g)$  is self-dual and Einstein,

$$I^{\text{ren}} = 12\pi^2 \eta(\gamma) + (8\pi^2 \chi(M) - 12\pi^2 \sigma(M)), \quad (5.3)$$

while  $I^{\text{ren}} = -12\pi^2 \eta(\gamma) + (8\pi^2 \chi(M) + 12\pi^2 \sigma(M))$ , if  $(M, g)$  is anti-self-dual Einstein.

This leads to the following result:

**Theorem 5.1.** *Let  $\mathcal{E}_{\text{sd}}$  be the moduli space of self-dual AH Einstein metrics on a 4-manifold  $M$ . Then  $I^{\text{ren}}: \mathcal{E}_{\text{sd}} \rightarrow \mathbb{R}$  is given by*

$$I^{\text{ren}} = 12\pi^2 \eta + c_M, \quad (5.4)$$

where  $c_M = 8\pi^2 \chi(M) - 12\pi^2 \sigma(M)$  is topological.

The stress-energy tensor  $g_{(3)}$  at a self-dual Einstein metric is given by

$$dI^{\text{ren}} = -\frac{3}{2}g_{(3)} = 12\pi^2 d\eta = -\frac{1}{2} * d \text{Ric} . \quad (5.5)$$

*Proof.* Both (5.4) and (5.5) follow immediately from (5.3). In (5.5), Ric is viewed as a 1-form with values in  $T(\partial M)$  and  $*$  is the Hodge  $*$ -operator. The formula for  $d\eta$  comes from the original work of Chern–Simons [18], cf. also [3] for more details.  $\square$

Similarly on the moduli space of anti-self-dual AH Einstein metrics  $\mathcal{E}_{\text{asd}}$ ,  $I^{\text{ren}} = -12\pi^2 \eta + c'_M$ , and  $dI^{\text{ren}} = -12\pi^2 d\eta = \frac{1}{2} * d \text{Ric}$ .

Thus, on the moduli spaces  $\mathcal{E}_{\text{sd}}$  or  $\mathcal{E}_{\text{asd}}$ , the renormalized action is explicitly computable from the *global* geometry of the boundary metric  $(\partial M, \gamma)$ , while the stress-energy is *locally computable* on  $(\partial M, \gamma)$ .

**Remark 5.2.** This exact identification of the renormalized gravitational action and its stress-energy tensor on  $\mathcal{E}_{\text{sd}}$  or  $\mathcal{E}_{\text{asd}}$  appears to closely resemble the identification of the renormalized action with the Liouville action on the boundary in dimensions  $2 + 1$ , cf. [39], [20]. It would be very interesting to pursue this analogy further.

The discussion above suggests that a self-dual (or anti-self-dual) AH Einstein metric should be uniquely determined by its boundary metric  $\gamma$ . This is true at least for real-analytic boundary data,  $\gamma \in C^\omega$ , as proved by LeBrun [40], using twistor methods. A more elementary proof, using just the Cauchy–Kovalevsky theorem, was given recently in [6]. This uniqueness implies that the topology of the bulk manifold  $M$  is determined by the boundary data  $(\partial M, \gamma)$ , up to covering spaces, i.e. any two self-dual AH Einstein metrics  $(M_1, g_1), (M_2, g_2)$  with the same boundary data are locally isometric; in particular they are isometric in some covering space.

Similarly, in analogy to the discussion following (1.9), [40] or [6] imply that any metric  $\gamma \in C^\omega(\partial M)$  is the boundary metric of a self-dual or anti-self-dual AH Einstein metric  $g$ , defined on a thickening  $\partial M \times [0, \varepsilon)$  of  $\partial M$ . Of course, for general  $\gamma$ , this metric will not extend to a smooth Einstein metric on a compact manifold  $M$  with boundary  $\partial M$ .

For example, consider the boundary  $S^2 \times S^1$ , with boundary metric the round conformally flat product metric. This bounds a self-dual AH Einstein metric in a neighborhood of  $S^2 \times S^1$ , and uniqueness implies that this metric must be the hyperbolic metric on  $B^3 \times S^1$ , given by  $H^4(-1)/\mathbb{Z}$ . The AdS–Schwarzschild metric  $S^2 \times \mathbb{R}^2$  is thus of course not self-dual.

It would be interesting to know if the moduli space  $\mathcal{E}_{\text{sd}}$  has the structure of an infinite dimensional manifold, as is the case with  $\mathcal{E}$  itself. Noteworthy in this respect is a result of Biquard [14] that in a neighborhood of the hyperbolic metric on  $B^4$ , the spaces  $\mathcal{E}_{\text{sd}}$  and  $\mathcal{E}_{\text{asd}}$  are smooth infinite dimensional manifolds, which intersect transversally at the hyperbolic metric  $g_{-1}$ . In particular, any metric  $g \in \mathcal{E}$  near  $g_{-1}$  can be uniquely written as a sum  $g = g^+ + g_{-1} + g^-$ , where  $g^+ + g_{-1}$  is self-dual and  $g_{-1} + g^-$  is

anti-self-dual. It seems that this result should be useful in understanding the Dirichlet-Neumann correspondence (1.9) near  $g_{-1}$ .

There are a number of interesting and explicit or semi-explicit examples of self-dual AH Einstein metrics. Thus, the AdS–Taub–NUT [32] or Pedersen metrics [44] are self-dual on  $B^4$ , with boundary metric a Berger sphere (the 3-sphere squashed along the  $S^1$  fibers of the Hopf fibration). More generally, Hitchin [36] has constructed self-dual AH Einstein metrics on  $B^4$  with conformal infinity any left-invariant metric on  $SU(2) = S^3$ . In addition, LeBrun [41] has proved the existence of an infinite dimensional family of self-dual AH Einstein metrics on  $B^4$ . More recently, Calderbank and Singer [17] have constructed families of self-dual AH Einstein metrics on resolutions of orbifolds  $\mathbb{C}^2/\mathbb{Z}_k$  having negative Chern class. This gives 4-manifolds with arbitrarily large Betti number  $b_2$  for which  $\mathcal{E}_{\text{sd}} \neq \emptyset$ .

## 6 Continuation to de Sitter and self-similar vacuum space-times

In this section we discuss the continuation of AH Einstein metrics to de Sitter-type Lorentz metrics, and the possibility of constructing global self-similar vacuum solutions of the Einstein equations in higher dimensions. The basic model for this picture is the decomposition of Minkowski space-time  $(\mathbb{R}^4, \eta)$  into foliations by hyperbolic metrics in the interior of the past and future light cones of a point  $\{0\}$ , and foliations by deSitter metrics in the exterior of the light cone. These foliations are of course given as the level sets of the distance function to  $\{0\}$ . Some aspects of this work have previously appeared in [8], see also [46] for a formal treatment of some of these issues.

Let  $M$  be any compact  $(n + 1)$  manifold with boundary  $\partial M$  and let  $g$  be any AH Einstein metric on  $M$ , with boundary metric  $(\partial M, \gamma)$ , with respect to a geodesic defining function  $\rho$ . As is well known, and observed by Fefferman–Graham [27] in their original work on the subject,  $(M, g)$  then generates a vacuum solution to the Einstein equations on  $\mathcal{M} = \mathbb{R}^+ \times M$  given by

$$\mathfrak{g} = -d\tau^2 + \tau^2 g, \quad (6.1)$$

where  $\tau \in (0, \infty)$ . This is a Lorentzian cone metric on the Riemannian metric  $g$ , and satisfies the vacuum equations

$$\text{Ric}_{\mathfrak{g}} = 0. \quad (6.2)$$

At least in a neighborhood of  $\mathbb{R}^+ \times \partial M$ , by (1.3), the metric (6.1) may be rewritten in the form

$$\mathfrak{g} = -d\tau^2 + \left(\frac{\tau}{\rho}\right)^2 (d\rho^2 + g_\rho). \quad (6.3)$$

The space-time  $(\mathcal{M}, \mathfrak{g})$  is globally hyperbolic, with Cauchy surface given by  $M$  and Cauchy data  $(g, K) = (g, g)$ . The time evolution with respect to the time parameter  $\tau$  is given by trivial rescalings of the time 1 spatial metric  $(M, g)$ . When  $(M, g) = (\mathbb{R}^3, g_{-1})$  is the Poincaré metric on the 3-ball,  $(\mathcal{M}, \mathfrak{g})$  is the interior of the past (or future) light cone of a point  $\{0\}$  in Minkowski space-time  $\mathbb{R}^4$  (also called the Milne universe). Similarly, for any  $(M, g)$  with  $g \in \mathcal{E}$ , the vacuum solution  $(\mathcal{M}, \mathfrak{g})$  is the interior of the past (or future) light cone  $(H, \gamma_0)$ , where  $H = \mathbb{R}^+ \times \partial M$  and the degenerate metric  $\gamma_0$  on  $H$  is given by

$$\gamma_0 = v^2 \gamma,$$

with  $v \in \mathbb{R}^+$  given by  $v = \tau/\rho$ . Thus,  $(H, \gamma_0)$  is the smooth Cauchy horizon for the space-time  $(\mathcal{M}, \mathfrak{g})$ . To be definite, we choose  $(\mathcal{M}, \mathfrak{g})$  to be the interior of the past light cone of the vertex  $\{0\} = \{v = 0\}$ , and will later set  $(\mathcal{M}, \mathfrak{g}) = (\mathcal{M}^-, \mathfrak{g}^-)$  and  $H = H^-$ .

In general, the metric  $\mathfrak{g}$  is not  $C^\infty$  up to the Cauchy horizon  $H$ . Thus, under the change of variables  $(\tau, \rho) \rightarrow (v, x)$ , with  $\rho = \sqrt{x}$ , (6.3) becomes

$$\mathfrak{g} = -x dv^2 - v dv dx + v^2 g_{\sqrt{x}}, \quad (6.4)$$

and for  $g_{\sqrt{x}}$  one has the Fefferman–Graham expansion

$$g_{\sqrt{x}} = \gamma + x g_{(2)} + \cdots + x^{n/2} g_{(n)} + \frac{1}{2} x^{n/2} \log x h + \cdots. \quad (6.5)$$

Hence, if  $n$  is odd, the metric  $\mathfrak{g}$  is  $C^{n/2}$  up to the horizon  $H$ , while if  $n$  is even,  $\mathfrak{g}$  is  $C^{n/2-\varepsilon}$  up to  $H$ . This degree of smoothness cannot be improved by passing to other coordinate systems; only in very rare instances where  $g_{(n)} = 0$  when  $n$  is odd, or  $h = 0$  when  $n$  is even, will  $\mathfrak{g}$  be  $C^\infty$  up to  $H$ .

Suppose first that  $n = 3$  and let  $\gamma$  be a real-analytic metric on  $\partial M$ . Then by [6], the compactification  $\bar{g} = \rho^2 g$  of  $(M, g)$  is also real-analytic on  $\bar{M}$ , and so the Fefferman–Graham expansion (1.4) converges to  $g_\rho$ . Thus, the curve  $g_\rho$  can be extended past  $\rho = 0$  to purely imaginary values of  $\rho$ . This corresponds to replacing  $\sqrt{x}$ ,  $x > 0$ , by  $-\sqrt{|x|}$ ,  $x < 0$  (i.e.  $\rho \rightarrow i\rho$ ) and so gives the curve

$$g_\rho^{\text{ext}} = \gamma - \rho^2 g_{(2)} - \rho^3 g_{(3)} + \rho^4 g_{(4)} + \cdots, \quad (6.6)$$

obtained from the expansion for  $g_\rho$  by replacing  $\rho$  by  $i\rho$  and dropping the  $i$  coefficients. Of course one could also continue the Fefferman–Graham expansion into the region  $\rho < 0$ ; this would give an AH Riemannian Einstein metric on “the other side” of  $\partial M$ , defined at least in some neighborhood of  $\partial M$ . However, this extension will not be of concern here.

Thus, although the metric  $(\mathcal{M}, \mathfrak{g})$  is only  $C^{3/2}$  at  $H$ , it extends via (6.6) and (6.4) across the horizon  $H$  into the exterior of the light cone. Returning to the original variables  $(\tau, \rho)$  in (6.3) then gives the metric

$$\mathfrak{g}^{\text{ext}} = d\tau^2 + \left(\frac{\tau}{\rho}\right)^2 (-d\rho^2 + g_\rho^{\text{ext}}). \quad (6.7)$$

Formally, this is obtained from  $g = g^-$  by the replacement  $\tau \rightarrow i\tau$ ,  $\rho \rightarrow i\rho$ , interchanging a spatial and time direction. This gives an extension of the metric  $g$  into a region  $\mathcal{M}^{\text{ext}}$  exterior to the light cone  $H$ , defined for all  $\tau \in (0, \infty)$ ,  $\rho \in [0, \varepsilon)$ , for some  $\varepsilon > 0$ . The metric  $g^{\text{ext}}$  is  $C^\omega$  where  $\rho \neq 0$ , but is only  $C^{3/2}$  up to  $H$  where  $\rho = 0$ .

The metric  $g^{\text{ext}}$  is also a Lorentzian cone metric, now however with a space-like self-similarity in  $\tau$  in place of the previous time-like self-similarity. The slices  $\tau = \text{const}$  are all homothetic, and are Lorentzian metrics of the form

$$g^{\text{dS}} = \left(\frac{1}{\rho}\right)^2 (-d\rho^2 + g_\rho^{\text{ext}}). \quad (6.8)$$

The metric  $g^{\text{dS}}$  is a solution to the vacuum Einstein equations with a positive cosmological constant  $\Lambda = \frac{1}{2}n(n-1)$ , i.e. when  $n = 3$ ,

$$\text{Ric}_{g^{\text{dS}}} = 3g^{\text{dS}}, \quad (6.9)$$

and so  $g^{\text{dS}}$  is a deSitter-type (dS) space-time (just as the initial metric  $g$  is of anti-deSitter type).

Exactly the same discussion holds in all dimensions. Thus, suppose again that  $\gamma \in C^\omega(\partial M)$ . As noted in §1, it follows from the recent regularity result of Chruściel et al. [19], that the compactification  $\bar{g} = \rho^2 g$  is  $C^\infty$  polyhomogeneous. Moreover, recent work of Kichenassamy [38] (cf. also Rendall [45]) implies the Fefferman–Graham expansion (1.4) or (1.6) converges to  $g_\rho$ , in both cases  $n$  even or  $n$  odd. Hence, exactly the same arguments as above hold for any  $n$ , and give a dS-type Einstein metric of the form (2.8) satisfying

$$\text{Ric}_{g^{\text{dS}}} = n g^{\text{dS}}, \quad (6.10)$$

with  $g_\rho^{\text{ext}}$  given by

$$g_\rho^{\text{ext}} = \gamma - \rho^2 g_{(2)} - \rho^3 g_{(3)} + \cdots \pm \rho^n g_{(n)} \pm \rho^n \log \rho h + \cdots. \quad (6.11)$$

The terms  $g_{(k)}$  are defined as in (1.8), where  $T = \bar{\nabla}\rho$  is the future-directed unit vector.

We summarize some of this discussion in the following result.

**Corollary 6.1.** *Let  $\gamma$  be a real-analytic metric on an  $n$ -manifold  $\partial M$ , and  $g_{(n)}$  a real-analytic symmetric bilinear form on  $\partial M$  satisfying the constraint conditions (1.5) or (1.7). Then there is a 1-1 correspondence between Riemannian AH Einstein metrics  $g$  with boundary metric  $\gamma$ , and deSitter-type Lorentzian Einstein metrics  $g^{\text{dS}}$  with past (or future) boundary metric  $\gamma$  given by (6.3)–(6.8).*

This correspondence thus gives a rigorous form of “Wick rotation” between these types of Einstein metrics. The Fefferman–Graham expansion holds for Einstein metrics of any signature (again as observed in [27]). In the correspondence between AH



and dS Einstein metrics, one has

$$g_{(k)}^{AH} = \pm g_{(k)}^{dS}, \quad (6.12)$$

where  $+$  occurs if  $k \equiv 0, 1 \pmod{4}$ , while  $-$  occurs if  $k \equiv 2, 3 \pmod{4}$ .

In dimension 4, the formulas (1.20) and (1.19) for the renormalized action and its variation also have analogues for dS space-times. Thus, let  $(\mathcal{S}, g)$  be a solution of (6.10) which is asymptotically simple, in that  $(\mathcal{S}, g)$  has a smooth past and future conformal infinity  $(\mathcal{I}^-, \gamma^-)$  and  $(\mathcal{I}^+, \gamma^+)$ . In particular,  $\mathcal{S}$  is geodesically complete and globally hyperbolic with compact Cauchy surface  $\Sigma$ , a 3-manifold diffeomorphic to  $\mathcal{I}^-$  and  $\mathcal{I}^+$ . In the following, we will forgo the exact determination of signs, which are best computed on an example; note that the Einstein–Hilbert action (1.11) is usually replaced by its negative for Lorentzian metrics.

**Proposition 6.2.** *Let  $(\mathcal{S}, g)$  be a 4-dimensional asymptotically simple vacuum dS space-time. Then*

$$\pm I^{\text{ren}} = \int_{\mathcal{S}} |W|^2 dV, \quad (6.13)$$

where  $|W|^2 = W_{ijkl} W^{ijkl}$  and

$$\pm dI^{\text{ren}} = g_{(3)}^+ - g_{(3)}^-, \quad (6.14)$$

where the terms are taken with respect to the past unit normal.

Since the metric  $g$  is Lorentzian, note that  $|W|^2$  is not a priori non-negative.

*Proof.* The proof of (6.13) is exactly the same as the proof of (1.20) in [3], using the Lorentz version of the Chern–Gauss–Bonnet theorem, cf. [1] for example, in place of the Riemannian version. Since  $\mathcal{S} = \mathbb{R} \times \Sigma$  topologically, where  $\Sigma$  is a closed 3-manifold,  $\chi(\mathcal{S}) = 0$ . Similarly, the proof of (1.19) in [3] holds equally well for Lorentzian metrics, and gives (6.14).  $\square$

Returning to the discussion preceding Corollary 6.1, the extended metric  $g$  on the enlarged space  $\mathcal{M}^- \cup \mathcal{M}^{\text{ext}}$  is still a solution to the vacuum Einstein equations (with  $\Lambda = 0$ ). This is clear if  $n > 4$ , since the metric is everywhere at least  $C^{5/2}$ . For  $n = 3, 4$ , the metric is not  $C^2$ , but is easily verified to still be a weak solution of the vacuum Einstein equations, i.e. it satisfies the equations (6.2) distributionally.

Now the initial AH Einstein metric  $(M, g)$  is global. It is natural to ask if the dS metric  $g^{dS}$  is also global; the formula (6.7) is only defined for  $\rho \in [0, \varepsilon)$ , for some  $\varepsilon > 0$ . In general, the answer is no. In fact, the work in §4 carries over to this setting almost identically, and gives the following result, proved independently by the author (unpublished) and Andersson–Galloway [11].

**Proposition 6.3.** *Let  $(\mathcal{S}, g)$  be an  $(n+1)$  dimensional globally hyperbolic space-time, with compact Cauchy surface  $\Sigma$ , which is  $C^3$  conformally compact to the past, so that*

past conformal infinity  $(\mathcal{I}^-, \gamma)$  is  $C^3$ . Suppose  $(\mathcal{S}, \mathfrak{g})$  satisfies the strong energy and decay conditions

$$(\text{Ric}_{\mathfrak{g}} - n\mathfrak{g})(T, T) \geq 0 \quad \text{and} \quad |(\text{Ric}_{\mathfrak{g}} - n\mathfrak{g})(T, T)| = o(\rho^2) \quad (6.15)$$

for  $T$  time-like. Let  $\gamma$  be a representative for  $[\gamma]$  with constant scalar curvature  $R_\gamma$ . If  $R_\gamma < 0$ , then

$$\rho^2(x) \leq 4n(n-1)/|R_\gamma|, \quad (6.16)$$

where  $\rho$  is the geodesic defining function associated to  $(\mathcal{I}^-, \gamma)$ .

In particular, any time-like geodesic in  $\mathcal{S}$  is future incomplete, and no Cauchy surface  $\Sigma_\rho$  exists, even partially, for  $\rho^2 > 4n(n-1)/|R_\gamma|$ , so that  $\mathcal{I}^+ = \emptyset$ .

*Proof.* The proof is essentially identical to that of Theorem 4.1. Let  $\bar{\mathfrak{g}} = \rho^2 \mathfrak{g}$  be the  $C^2$  geodesic compactification determined by the data  $(\mathcal{I}^-, \gamma)$ ; as before the computations below are with respect to  $\bar{\mathfrak{g}}$ . The equation (4.3) holds for Lorentzian metrics (where it is known as the Raychaudhuri equation). The vector field  $T = \bar{\nabla} \rho$  is now a unit time-like vector field, so  $\mathfrak{g}(T, T) = -1$ . This has the implication that  $H = -\bar{\Delta} \rho$  while

$$\overline{\text{Ric}} = -(n-1) \frac{\bar{D}^2 \rho}{\rho} - \frac{\bar{\Delta} \rho}{\rho} \bar{\mathfrak{g}} + (\text{Ric}_{\mathfrak{g}} - n\mathfrak{g}) \geq -(n-1) \frac{\bar{D}^2 \rho}{\rho} - \frac{\bar{\Delta} \rho}{\rho} \bar{\mathfrak{g}},$$

and

$$\bar{R} = -2n \frac{\bar{\Delta} \rho}{\rho} + (R - n(n+1))/\rho^2 \geq -2n \frac{\bar{\Delta} \rho}{\rho}.$$

In particular,  $\overline{\text{Ric}}(T, T) = \frac{\bar{\Delta} \rho}{\rho} + (\text{Ric}_{\mathfrak{g}} - n\mathfrak{g})(T, T)/\rho^2$ . The same arguments as in (4.4)–(4.7) then give

$$\phi' \leq -\frac{1}{n} \rho \phi^2, \quad (6.17)$$

where again  $\phi = -\bar{\Delta} \rho / \rho$ . The formula (4.8) also holds, and hence  $R_\gamma < 0$  implies  $\phi(0) < 0$ . Integrating (6.17) as before gives (6.16).  $\square$

It is straightforward to extend Proposition 6.3 to the situation where  $R_\gamma = 0$ . As in the case of Theorem 4.1,  $(\mathcal{S}, \mathfrak{g})$  has  $\mathcal{I}^+ = \emptyset$  and no time-like geodesic is future-complete unless  $(\mathcal{S}, \mathfrak{g})$  is isometric to

$$\mathfrak{g} = -dt^2 + e^{2t} g_{N^n},$$

where  $g_{N^n}$  is Ricci-flat, cf. [11] for further details.

Proposition 6.3 implies that dS space-times  $(\mathcal{S}, \mathfrak{g})$  satisfying the strong energy and decay conditions (6.15) cannot be geodesically complete if  $R_\gamma < 0$ , and have at most one component of conformal infinity. This exhibits the role of the hypothesis  $R_\gamma > 0$  in a more drastic way than the AH case.

We are interested in understanding when the vacuum dS metric  $g^{\text{dS}}$  constructed in (6.8) is also complete to the future, and has a smooth future conformal infinity  $\mathcal{I}^+$ . In dimensions  $n + 1 > 4$ , it is an interesting open problem to find sufficient conditions guaranteeing the existence of complete asymptotically simple vacuum dS space-times. However, in dimension 4, a basic result of H. Friedrich does give global existence of dS vacuum solutions, for small perturbations of the exact deSitter metric.

**Theorem 6.4** ([28]). *Let  $\gamma$  be a smooth metric on  $S^3$  and  $\sigma$  be a smooth transverse-traceless symmetric bilinear form on  $S^3$ . Suppose that  $\gamma$  is sufficiently close to the round metric  $g_{+1}$  on  $S^3$  and  $|\sigma|$  is sufficiently small (measured with respect to  $\gamma$ ). Then there exists a unique asymptotically simple vacuum dS space-time  $(\mathcal{S}, g)$ ,  $\mathcal{S} = S^3 \times \mathbb{R}$  with smooth conformal compactification  $\bar{\mathcal{S}} = \mathcal{S} \cup \mathcal{I}^- \cup \mathcal{I}^+$  for which the Fefferman–Graham expansion satisfies  $g_{(0)} = \gamma$  and  $g_{(3)} = \sigma$  on  $\mathcal{I}^-$ . If  $\gamma$  and  $\sigma$  are  $C^\omega$ , then  $(\mathcal{S}, g)$  is  $C^\omega$  conformally compact.*

Note that, in contrast to the AH or AdS situation,  $g_{(0)}$  and  $g_{(3)}$  are freely specifiable on  $\mathcal{I}^-$ , subject to smallness conditions. An alternate version of the result should allow one to freely specify  $g_{(0)}$  on  $\mathcal{I}^+$  and  $\mathcal{I}^-$ , provided they are both close to the round metric on  $S^3$ ; this remains to be proved however.

**Remark 6.5.** This result is an exact analogue of the result of Graham–Lee [30] on AH Einstein perturbations of the Poincaré metric on the ball  $B^{n+1}$  (since Friedrich’s result predates that of Graham–Lee, the opposite statement is more accurate). It would be very interesting if a higher dimensional analogue of Friedrich’s result could be proved, as in the Graham–Lee theorem.

We may now apply this result to the “initial” AH Einstein metric  $(M, g)$ ,  $g = g^-$ . Thus, on  $M = B^4$ , let  $g^-$  be an AH Einstein metric with  $C^\omega$  boundary metric  $\gamma^-$  close to the round metric  $\gamma_{+1}$  on  $S^3$ , (so  $g^-$  is close to the Poincaré metric on  $B^4$ ). The metric  $g^-$  determines the terms  $\gamma = g_{(0)} = g_{(0)}^{AH}$  and  $g_{(3)}^{AH}$  in the Fefferman–Graham expansion. Let  $g^{\text{dS}}$  be the unique vacuum dS solution given by Friedrich’s theorem satisfying

$$(g_{(0)}^{\text{dS}})_{\mathcal{I}^-} = g_{(0)}^{AH} \quad \text{and} \quad (g_{(3)}^{\text{dS}})_{\mathcal{I}^-} = -g_{(3)}^{AH}, \quad (6.18)$$

where  $g_{(3)}^{\text{dS}}$  is defined as in (1.8) with respect to the future normal  $T = \bar{\nabla}\rho$ . Thus, the stress-energy tensors of  $g^-$  and  $g^{\text{dS}}$  cancel at  $\mathcal{I}^-$ .

The vacuum dS solution  $g^{\text{dS}}$  is globally defined, and has a  $C^\omega$  compactification to  $\mathcal{I}^-$  and  $\mathcal{I}^+$ . Let  $(g_{(0)}^{\text{dS}})_{\mathcal{I}^+}$  and  $(g_{(3)}^{\text{dS}})_{\mathcal{I}^+}$  be the boundary metric and stress-energy tensor of  $g^{\text{dS}}$  at future conformal infinity  $\mathcal{I}^+$ . Then  $(g_{(0)}^{\text{dS}})_{\mathcal{I}^+}$  is close to the round metric  $\gamma_{+1}$  on  $S^3$ , while  $(g_{(3)}^{\text{dS}})_{\mathcal{I}^+}$  is close to 0, and both are real-analytic. By the Graham–Lee theorem [30], there is an AH Einstein metric  $g^+$  on  $B^4$  with boundary metric  $\gamma_+ = (g_{(0)}^{\text{dS}})_{\mathcal{I}^+}$ , and by boundary regularity [6],  $g^+$  has a real-analytic compactification.

Thus, we have constructed a global  $4 + 1$  dimensional space-time

$$(\mathcal{M}, \mathfrak{g}) = (\mathcal{M}^-, g^-) \cup (\mathcal{M}^{\text{ext}}, \mathfrak{g}^{\text{ext}}) \cup (\mathcal{M}^+, g^+). \quad (6.19)$$

This space-time is globally self-similar, with

$$\mathcal{L}_{\nabla\tau}\mathfrak{g} = 2\mathfrak{g}, \quad (6.20)$$

with a singularity at the vertex  $\{0\}$ . The metric  $\mathfrak{g}$  is  $C^\omega$  off the null cone  $H = H^- \cup H^+$  and is  $C^{3/2}$  across the null-cone away from  $\{0\}$ .

In general however, it is not clear if  $(\mathcal{M}, \mathfrak{g})$  is a vacuum space-time. The stress-energy tensor  $g_{(3)}^+$  of  $g^+$  is globally determined by the boundary metric  $\gamma^+$ , and there is no apriori reason that one should have

$$g_{(3)}^+ = -(g_{(3)}^{\text{dS}})_{\mathcal{I}^+}, \quad (6.21)$$

as given by construction at  $\mathcal{I}^-$ . Thus, there may be an effective stress-energy of the gravitational field along the future light cone  $H^+$  of  $\{0\}$ . Of course the vacuum equations (6.9) are satisfied everywhere off  $H^+$ .

It is of interest to understand if there exist non-trivial situations where (6.21) does hold, or to prove that it cannot hold. If (6.21) holds, then  $(\mathcal{M}, \mathfrak{g})$  is a globally defined self-similar vacuum solution, with an isolated (naked) singularity at  $\{0\}$ . Of course in  $3 + 1$  dimensions the only such space-time is empty Minkowski space  $(\mathbb{R}^4, \eta)$ .

We examine this issue on a particular family of examples.

**Example 6.6.** Let  $g^-$  be the AdS–Taub–NUT metric on  $B^4$ , cf. [32] for example (also called the Pedersen metric [44]), given by

$$g^- = \frac{E(r^2 - 1)}{F(r)} dr^2 + \frac{EF(r)}{(r^2 - 1)} \theta_1^2 + \frac{E(r^2 - 1)}{4} g_{S^2(1)}, \quad (6.22)$$

where  $E \in (0, \infty)$  is any constant,  $r \geq 1$ , and

$$F(r) = Er^4 + (4 - 6E)r^2 + (8E - 8)r + 4 - 3E. \quad (6.23)$$

The length of the  $S^1$  parametrized by  $\theta_1$  is  $2\pi$ . This metric is self-dual Einstein and has conformal infinity  $\gamma^-$  given by the Berger (or squashed) sphere with  $S^1$  fibers of length  $\beta = 2\pi E^{1/2}$  over  $S^2(1)$ . Clearly  $\gamma^-$  is  $C^\omega$ , as is the geodesic compactification with boundary metric  $\gamma^-$ . Since  $g^-$  is self-dual, the stress-energy tensor  $g_{(3)}$  is given by (5.5). When  $E = 1$ ,  $g^-$  is the Poincaré metric.

The deSitter continuation of  $g^-$  is the dS–Taub–NUT metric on  $\mathbb{R} \times S^3$ , cf. [15] for instance, given by

$$\mathfrak{g}^{\text{dS}} = -\frac{E(\tau^2 + 1)}{A(\tau)} d\tau^2 + \frac{EA(\tau)}{(\tau^2 + 1)} \theta_1^2 + \frac{E(\tau^2 + 1)}{4} g_{S^2(1)}, \quad (6.24)$$

where  $\tau \in (-\infty, \infty)$  and

$$A(\tau) = E\tau^4 - (4 - 6E)\tau^2 - (8E - 8)\tau + 4 - 3E. \quad (6.25)$$

Again when  $E = 1$ ,  $g^{\text{dS}}$  is the (exact) deSitter metric. For  $g^{\text{dS}}$  to be complete and globally hyperbolic, without singularities, one needs  $A(\tau) > 0$  for all  $\tau$ . A lengthy but straightforward calculation shows this is the case exactly when

$$E \in \left( \frac{2}{3}, \frac{1}{3}(2 + \sqrt{3}) \right). \quad (6.26)$$

Suppose then  $E$  satisfies (6.26). By construction, the metrics  $g^-$  and  $g^{\text{dS}}$  satisfy (6.18). Observe from the explicit form of (6.24) that

$$\gamma^- = \gamma^+. \quad (6.27)$$

Thus, even though  $g^{\text{dS}}$  is not time-symmetric when  $E \neq 1$ , there is no gravitational scattering from past to future conformal infinity, in the sense that  $\mathcal{I}^-$  is isometric to  $\mathcal{I}^+$ . However, further computation shows that (6.21) does not hold; instead one has  $g_{(3)}^+ = (g_{(3)}^{\text{dS}})_{\mathcal{I}^+}$ , so that the full metric  $g$  has an effective stress-energy tensor on the future null cone of  $\{0\}$ . We note however that there is an AH Taub–NUT metric satisfying (6.21) which has an isolated conical (nut) singularity at the origin of  $B^4$ ; in this case, the formula (6.23) is replaced by a more general formula allowing two independent parameters, the mass and nut charge, in place of the one parameter  $E$ , cf. [24], [25]. If one fills in  $H^+$  with such a metric, then the effective stress-energy tensor of  $(\mathcal{M}, g)$  is located on the future world line of the singularity  $\{0\}$ .

Returning to the discussion of the de Sitter metrics (6.24), at the extremal values  $E_- = \frac{2}{3}$  and  $E_+ = \frac{1}{3}(2 + \sqrt{3})$ , the metric  $g^{\text{dS}}$  is still complete and globally hyperbolic. However, at these values,  $g^{\text{dS}}$  is not in the space of metrics with smooth  $\mathcal{I}^+$  and  $\mathcal{I}^-$ ; instead, it is in the boundary of this space. To explain this, let  $\tau_- = -1$ , and  $\tau_+ = 2 - \sqrt{3}$ . Then at  $E_{\pm}$ ,  $A(\tau) \geq 0$ , with  $A(\tau) = 0$  exactly at  $\tau_{\pm}$ . Each metric  $g^{\text{dS}}$  breaks up into a pair of complete, globally hyperbolic metrics,  $g_p^{\text{dS}}$  and  $g_f^{\text{dS}}$  parametrized on  $(-\infty, \tau_{\pm})$  and  $(\tau_{\pm}, \infty)$  respectively. The metric  $g_p^{\text{dS}}$  has a smooth past conformal infinity  $\mathcal{I}^-$ , but  $\mathcal{I}^+ = \emptyset$ , while  $g_f^{\text{dS}}$  has a smooth future conformal infinity  $\mathcal{I}^+$  but  $\mathcal{I}^- = \emptyset$ . These metrics correspond to degenerate black hole metrics, and are analogous, in a dual sense, to the situation in Remark 2.3.

For  $E$  outside the range (6.26), the function  $A(\tau)$  changes sign, and the metrics  $g^{\text{dS}}$  develop closed time-like curves, as with the behavior of the  $\Lambda = 0$  Taub–NUT metrics

It is also straightforward to compute that  $R_{\gamma^-} > 0$  exactly for  $E$  in the range  $E \in (0, 4)$ . This shows that the converse of Proposition 6.3 does not hold, i.e. the condition  $R_{\gamma} > 0$  is not sufficient to imply that a vacuum dS solution with smooth  $\mathcal{I}^-$  is complete to the future.

It would be very interesting to generalize this example. For instance, can the same construction of globally self-similar almost-vacuum solutions be carried out for general AdS- and dS-Bianchi IX space-times, which have conformal infinity a general  $SU(2)$  invariant metric on  $S^3$ ? The AdS-Bianchi IX metrics are self-dual, and have been described in detail by Hitchin [36]. Is the relation (6.27), related to self-duality?

## References

- [1] L. Alty, The generalized Gauss–Bonnet–Chern theorem, *J. Math. Phys.* 36 (1995), 3094–3105.
- [2] M. Anderson, The  $L^2$  structure of moduli spaces of Einstein metrics on 4-manifolds, *Geom. Funct. Anal.* 2 (1992), 29–89.
- [3] M. Anderson,  $L^2$  curvature and volume renormalization for AHE metrics on 4-manifolds, *Math. Res. Lett.* 8 (2001), 171–188.
- [4] M. Anderson, Boundary regularity, uniqueness and non-uniqueness for AH Einstein metrics on 4-manifolds, *Adv. Math.* 179 (2003), 205–249.
- [5] M. Anderson, Einstein metrics with prescribed conformal infinity on 4-manifolds, preprint, May 01/Feb 04, math.DG/0105243.
- [6] M. Anderson, Some results on the structure of conformally compact Einstein metrics, preprint, Feb. 04, math.DG/0402198.
- [7] M. Anderson, Dehn filling and Einstein metrics in higher dimensions, preprint, March 03, math.DG/0303260.
- [8] M. Anderson, Remarks on evolution of spacetimes in  $3 + 1$  and  $4 + 1$  dimensions, *Classical Quantum Gravity* 18 (2001), 5199–5209.
- [9] M. Anderson, S. Carlip, J. Ratcliffe, S. Surya and S. Tschantz, Peaks in the Hartle–Hawking wave function from sums over topologies, *Classical Quantum Gravity* 21 (2004), 729–741.
- [10] M. Anderson, P. Chruściel and E. Delay, Non-trivial static, geodesically complete vacuum space-times with a negative cosmological constant, *J. High Energy Phys.* 10 (2002), 063.
- [11] L. Andersson and G. Galloway, dS/CFT and spacetime topology, *Adv. Theor. Math. Phys.* 6 (2003), 307–327.
- [12] V. Balasubramanian and P. Kraus, A stress tensor for anti-de Sitter gravity, *Comm. Math. Phys.* 208 (1999), 413–428.
- [13] O. Biquard, Métriques d’Einstein asymptotiquement symétriques, *Astérisque* 265 (2000).
- [14] O. Biquard, Métriques autoduales sur la boule, *Invent. Math.* 148 (2002), 545–607.
- [15] D. Brill and F. Flaherty, Maximizing properties of extremal surfaces in General Relativity, *Ann. Inst. H. Poincaré Sect. A (N.S.)* 28 (1978), 335–347.
- [16] M. Cai and G. Galloway, Boundaries of zero scalar curvature in the AdS/CFT correspondence, *Adv. Theor. Math. Phys.* 3 (1999), 1769–1783.
- [17] D. Calderbank and M. Singer, Einstein metrics and complex singularities, *Invent. Math.* 156 (2004), 405–443.
- [18] S. S. Chern and J. Simons, Characteristic forms and geometric invariants, *Ann. of Math.* 99 (1974), 48–69.
- [19] P. Chruściel, E. Delay, J. M. Lee and D. Skinner, Boundary regularity of conformally compact Einstein metrics, math.DG/0401386.

- [20] O. Coussaert, M. Henneaux and P. van Driel, The asymptotic dynamics of three-dimensional Einstein gravity with a negative cosmological constant, *Classical Quantum Gravity* 12 (1995), 2961–2966.
- [21] G. Craig, Dehn filling and asymptotically hyperbolic Einstein manifolds, Thesis, SUNY at Stony Brook, 2004, to appear.
- [22] S. deHaro, K. Skenderis and S. Solodukhin, Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence, *Comm. Math. Phys.* 217 (2001), 595–622.
- [23] R. Dijkgraaf, J. Maldacena, G. Moore and E. Verlinde, A black hole farey tail, hep-th/0005003.
- [24] R. Emparan, AdS/CFT duals of topological black holes and the entropy of zero-energy states, *J. High Energy Phys.* 06 (1999), 036.
- [25] R. Emparan, C. Johnson and R. Myers, Surface terms as counterterms in the AdS/CFT correspondence, *Phys. Rev. D* 60 (1999), 104001.
- [26] C. Epstein, An asymptotic volume formula for convex cocompact hyperbolic manifolds, in: Appendix A, The divisor of Selberg’s zeta function for Kleinian groups, by S. Patterson and P. Perry, *Duke Math. J.* 106 (2001), 370–379.
- [27] C. Fefferman and C. R. Graham, Conformal invariants, in: *Élie Cartan et les mathématiques d’aujourd’hui* (Lyon 1984), *Astérisque* (1985), Numéro hors-série, 95–116.
- [28] H. Friedrich, On the existence of  $n$ -geodesically complete or future complete solutions of Einstein’s field equations with smooth asymptotic structure, *Comm. Math. Phys.* 107 (1986), 587–609.
- [29] G. Galloway, S. Surya and E. Woolgar, A uniqueness theorem for the anti-de Sitter soliton, *Phys. Rev. Lett.* 88 (2002), 101102.
- [30] C. R. Graham and J. M. Lee, Einstein metrics with prescribed conformal infinity on the ball, *Adv. Math.* 87 (1991), 186–225.
- [31] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Gauge theory correlators from non-critical string theory, *Phys. Lett. B* 428 (1998), 105–114.
- [32] S. Hawking, C. Hunter and D. Page, Nut charge, anti-de Sitter space and entropy, *Phys. Rev. D* 59 (1999), 044033.
- [33] S. Hawking and D. Page, Thermodynamics of black holes in Anti-de Sitter space, *Comm. Math. Phys.* 87 (1983), 577–588.
- [34] M. Henningson and K. Skenderis, The holographic Weyl anomaly, *J. High Energy Phys.* 07 (1998), 023.
- [35] N. Hitchin, Einstein metrics and the eta-invariant, *Boll. Unione Mat. Ital. B* (7) 11 (1997), 95–105.
- [36] N. Hitchin, Twistor spaces, Einstein metrics and isomonodromic deformations, *J. Differential Geom.* 42 (1995), 30–112.
- [37] G. Horowitz and R. Myers, The AdS/CFT correspondence and a new positive energy conjecture for general relativity, *Phys. Rev. D* 59 (1999), 026005.

- [38] S. Kichenassamy, On a conjecture of Fefferman and Graham, *Adv. Math.* (2004), in press.
- [39] K. Krasnov, Holography and Riemann surfaces, *Adv. Theor. Math. Phys.* 4 (2000), 929–979.
- [40] C. LeBrun,  $\mathcal{H}$ -space with a cosmological constant, *Proc. Royal Soc. London Ser. A* 380 (1982), 171–185.
- [41] C. LeBrun, On complete quaternionic Kähler manifolds, *Duke Math. J.* 63 (1991), 723–743.
- [42] J. Maldacena, The large  $N$  limit of superconformal field theories and supergravity, *Adv. Theor. Math. Phys.* 2 (1998), 231–252.
- [43] R. Mazzeo and F. Pacard, Maskit combinations of Poincaré-Einstein metrics, math.DG/0211099.
- [44] H. Pedersen, Einstein metrics, spinning top motions and monopoles, *Math. Ann.* 274 (1986), 35–59.
- [45] A. Rendall, Asymptotics of solutions of the Einstein equations with positive cosmological constant, gr-qc/0312020.
- [46] K. Skenderis, Lecture notes on holographic renormalization, *Classical Quantum Gravity* 19 (2002), 5849–5876.
- [47] E. Witten, Anti De Sitter space and holography, *Adv. Theor. Math. Phys.* 2 (1998), 253–291.
- [48] E. Witten and S.-T. Yau, Connectedness of the boundary in the AdS/CFT correspondence, *Adv. Theor. Math. Phys.* 3 (1999), 1635–1655.





# Some aspects of the AdS/CFT correspondence

*Jan de Boer, Liat Maoz and Asad Naqvi*

*Instituut voor Theoretische Fysica*

*Valckenierstraat 65, 1018XE Amsterdam, The Netherlands*

*email: jdeboer@science.uva.nl, lmaoz@science.uva.nl*

*anaqvi@science.uva.nl*

**Abstract.** This is a very brief review of some aspects of the AdS/CFT correspondence with an emphasis on the role of the topology of the boundary and the meaning of the sum over bulk geometries.

## 1 Introduction

Since its incarnation in 1997 [47], [30], [59] (for a review see [1]), the AdS/CFT correspondence has been one of the prime subjects of interest in string theory. It provides a duality between a theory with quantum gravity in  $d$  dimensions and a field theory in  $d - 1$  dimensions. This is a rare example where we have a complete non-perturbative definition of string theory in a certain background, and quite amazingly it is equivalent to just an ordinary field theory. Strongly coupled string theory is equivalent to weakly coupled field theory and vice versa, and therefore the AdS/CFT correspondence can help understand the physics of strongly coupled gauge theories. At the same time, although Anti-de Sitter space has different asymptotics than Minkowski space or our universe (as far as we know), the properties of gravity at short distances should be somewhat independent of the asymptotic behavior of the space. Therefore the AdS/CFT correspondence should also be useful in understanding the puzzles associated to quantum gravity, in particular those associated with black hole creation and evaporation and information loss in black holes.

Though it is easy to say these words, to actually implement them in practice is not quite so straightforward. Given a certain manifold  $M$  on which the CFT lives, which can be either Lorentzian or Euclidean, the dual gravitational description involves a sum over all geometries whose conformal boundaries are equal to  $M$ . This by itself is a mathematical question, namely the classification of all solutions to the Einstein equations with a negative cosmological constant with given conformal boundary. However, this is not the full story, since we should really sum over all solutions of the string theory equations of motion with the right asymptotic behavior. String theory has various other fields in addition to the metric, and therefore this set can be larger than the set of

purely gravitational solutions. Furthermore, the string theory configurations that have to be summed over do not need to be weakly coupled everywhere and can in principle include stringy objects such as branes in the interior. The full classification of all such solutions is still in its infancy and a further understanding seems crucial in order to make progress in our understanding of quantum gravity.

With these motivations, we will here briefly summarize some of the known problems and solutions associated to finding the bulk solutions with a priori given asymptotia. We have no pretense of being complete and/or exhaustive, the main purpose of these notes is to provide some food for further thought.

There are in principle many cases to consider, and we organized them as follows. In Section 2 we consider solutions with Euclidean signature and a single connected boundary. In Section 3 we consider static solutions with Lorentzian signature and a single connected boundary. In Section 4 we discuss solutions with more than one boundary, and in Section 5 we briefly comment on the relation of all this with the puzzles associated to singularities and black holes in quantum gravity. In Section 6 we comment on the role of Chern–Simons theory in  $2 + 1$  dimensions; this is an especially interesting case since gravity has no propagating degrees of freedom in  $2 + 1$  dimensions. Finally, in Section 7 we mention some examples that involve time-dependent geometries, and in Section 8 some of the problems associated to extending all this to zero and positive cosmological constant are summarized.

## 2 Euclidean, single boundary setups

**AdS/CFT – the statement.** The original statement of AdS/CFT is a relationship between the partition function of string theory on  $\text{AdS} \times X$  geometry and that of a CFT living on the boundary of AdS:

$$Z_{\text{CFT}}(\partial M; \gamma) = \int \mathcal{D}\Phi_{\text{string}} e^{-S_{\text{SFT}}} \overset{\text{saddle point}}{\approx} \sum_i Z_{\text{string}}(M_i). \quad (1)$$

The CFT lives on  $\partial M$  which carries a metric in a fixed conformal class indicated by  $\gamma$ , and the LHS of (1) is the CFT partition function. We have schematically written the string path integral as an integral over string fields approaching  $\partial M$  on the boundary, and  $S_{\text{SFT}}$  represents the string field theory action. In going from this somewhat schematic expression to a more useful expression  $\sum_i Z_{\text{string}}(M_i)$ , we have made a saddle point approximation.  $M_i$  are backgrounds which satisfy the string equations of motion and have  $(\partial M; \gamma)$  as their conformal boundary. In most cases, the string coupling constant and the inverse curvature radius of the AdS space are free parameters and we can take them arbitrarily small. In this limit<sup>1</sup> we can replace the string theory partition function with the classical supergravity contribution, which is

---

<sup>1</sup>On the dual CFT side, the interpretation of this limit depends on the model, but generically involves taking a strong coupling limit, and also often involves sending the rank of a gauge group to infinity.

simply  $Z_{\text{Sugra}} = e^{-S(M_i)}$  where  $S(M_i)$  is the classical supergravity action evaluated on the solution  $M_i$ .

Notice the sum over different manifolds with the same conformal boundary. This plays a crucial role in the study of phase transitions in the boundary CFT. In a certain regime of parameters, generically one of the spaces  $M_i$  will dominate the sum on the right-hand side of (1). However, by varying the parameters, which of these  $M_i$ 's dominates may change, leading to phase transitions. The existence of a phase transition is perhaps surprising if the boundary  $\partial M$  is compact (as will mostly be the case in what we study below). However, in the large  $N$  limit, we can still have sharp phase transitions, even on a compact volume.

An important ill-understood feature of (1) is the precise relative normalization of the contributions on the right hand side. In the absence of a background independent formulation of string field theory it is not obvious how to compute these from first principles. The naive guess to take just the supergravity actions is incorrect in the example discussed in [22], but the microscopic origin of the relative normalization found in that paper is not clear.

In this section, we will discuss Euclidean setups, with the bulk space  $M$  being a solution to Einstein's equations with  $\Lambda < 0$ , which has a single boundary  $\partial M$ . We restrict ourselves to the case of a four dimensional bulk which is one of the best studied cases [3].

## 2.1 $\partial M = S^3, M = B^4$

This is the best understood example. Euclidean  $\text{AdS}_4$  can be described as the open unit ball  $B^4$ , with coordinates  $x_i$  such that  $\sum_{i=1}^4 x_i^2 < 1$  and the metric

$$ds^2 = \frac{4dx^2}{(1 - |x|^2)^2}. \quad (2)$$

This metric does not extend to the boundary at  $|x|^2 = 1$ . However, the metric can be extended to the boundary by defining a function  $f$  on the closure of  $M$  such that it has a simple zero at the boundary and is positive in the interior. Then, the metric  $d\tilde{s}^2 = f^2 ds^2$  extends to a metric on the boundary, but given a bulk metric, only the conformal structure of the boundary metric can be uniquely determined. Following a theorem of Graham and Lee, for every conformal structure on  $S^3$  sufficiently close to the standard one, there exists a metric on  $B^4$  with that  $S^3$  as a conformal boundary. Analogous statements hold for the scalar fields as well as the gauge fields-bulk fields are uniquely specified by their behavior on the boundary. This means that when we apply the AdS/CFT correspondence in this background, we can compute the correlation functions in the boundary theory by evaluating the bulk action for field configurations which asymptotically approach a given boundary data. On the right-hand side of (1), there is only one term in the summation.

## 2.2 $\partial M = S^1 \times S^2$ , $M = S^1 \times R^3$ and $M = R^2 \times S^2$

With the boundary  $S^1_\beta \times S^2$  (where  $S^1_\beta$  is a circle of radius  $\beta$ ), there are two known asymptotically AdS bulk solutions with this boundary. One is AdS itself (with topology  $S^1 \times R^3$ ), with the Euclidean time direction being a circle. This background is appropriate for the finite temperature bulk physics at temperature  $1/\beta$ . Another solution with the same boundary behavior is Euclidean AdS–Schwarzschild (this has topology  $R^2 \times S^2$ ). The boundary theory is a CFT at finite temperature.

It was shown by Witten [60] that the first solution dominates the partition function computation at low temperatures while the latter becomes dominant at high temperatures. This difference in behavior corresponds to the confinement–deconfinement phase transition in the field theory. This phase transition was first studied by Hawking and Page [35] who showed that above a critical temperature, thermal radiation is unstable to the formation of an AdS Schwarzschild black hole. There are, in fact, two black hole solutions, with different masses for a given value of  $\beta$  (the temperature). The smaller value of the masses leads to a black hole with a negative specific heat, which means that the black hole is unstable to decay as Hawking radiation. For the higher value of the masses, the Hawking radiation is in thermal equilibrium with the thermal radiation in the background.

## 2.3 $\partial M = T^3$ , $M = R^2 \times T^2$

These are the AdS toroidal black hole metrics, which are of the form

$$ds^2 = U^{-2}dr^2 + U^2d\theta^2 + r^2ds_{T^2}^2, \quad (3)$$

where  $U^2 = r^2 - \frac{2m}{r}$ . The conformal boundary of this space is  $S^1_\theta \times T^2 = T^3$ . Given a boundary metric, there are actually infinitely many different ways of filling in the bulk metric, each corresponding to a choice of one cycle in  $T^3$  which is ‘filled in’ to obtain a bulk solution. For details see [3]. The boundary theory is a finite temperature CFT on  $T^2$ . The multiple classical solutions perhaps correspond to different phases in this theory. Heuristically, the CFT partition function can be written as

$$Z_{\text{CFT}} \sim \sum_{g \in \text{SL}(3, \mathbb{Z})/H} \exp(-I(M_g)). \quad (4)$$

This expression should be taken with a large grain of salt. We do not really understand how to perform this sum here. In one lower dimension, for a two dimensional boundary, a similar summation was performed in [22] where the elliptic genus of the conformal field theory was computed by writing it as a sum over different asymptotically  $\text{AdS}_3 \times S^3$  bulk geometries (also see Section 5.3). The  $\text{AdS}_3$  string theory should reduce to a Chern–Simons theory at large distances. The calculation of the bulk partition function in this Chern–Simons theory is a state in the space of conformal blocks of the boundary theory and therefore transforms non-trivially under the modular group [32].

On the other hand, the string theory computation in [22] gives a modular invariant partition function. This apparent paradox is resolved [32] by the special appearance of a modular invariance restoring chiral “spectator boson” on the boundary.

## 2.4 AdS Taub–Bolt metrics

The AdS Taub–Bolt metrics are locally asymptotically AdS. The conformal boundary is an  $S^1$  bundle over  $S^2$ , with non-zero first Chern number. For vanishing first Chern number, the boundary is the product space  $S^1 \times S^2$  and the space is asymptotically AdS. This is one of the cases we discussed above. However, for non-vanishing first Chern number,  $k$ , the conformal boundary is a squashed  $S^3$  with  $|k|$  points identified along the  $S^1$ . These metrics have a  $U(1)$  isometry which acts on the  $S^1$  fiber in the natural way. For the AdS Taub–Bolt metric, the fixed point set of this isometry is two-dimensional (called a bolt). The line element is given by

$$ds^2 = -\frac{3}{4\Lambda} E \left[ \frac{F(r)}{E(r^2 - 1)} (d\tau + E^{1/2} \cos \theta d\phi)^2 + \frac{4(r^2 - 1)}{F(r)} dr^2 + (r^2 - 1)(d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (5)$$

with

$$F_{\text{Bolt}}(r) = Er^4 + (4 - 6E)r^2 + \left( -Es^3 + (6E - 4)s + \frac{3E - 4}{s} \right) r + 4 - 3E \quad (6)$$

and

$$E = \frac{2ks - 4}{3(s^2 - 1)}. \quad (7)$$

Here  $\Lambda < 0$  is the cosmological constant,  $\tau$  has period  $\beta = \frac{4\pi E^{1/2}}{k}$ ,  $s$  is an arbitrary parameter (the bolt is at  $r = s$ ) and  $k$  is the Chern number of the  $S^1$  bundle over  $S^2$ , which is the conformal boundary of this solution.  $|k|$  points on the  $S^1$  fiber are identified.

There is another class of closely related metrics for which the fixed point set of  $\frac{\partial}{\partial \tau}$  is just a point. These are the AdS Taub–NUT metrics. The line element for these metrics has the same form as (5) but the function  $F(r)$  is now given by

$$F_{\text{NUT}}(r) = Er^2 + (4 - 6E)r^2 + (8E - 8)r + 4 - 3E. \quad (8)$$

Now  $E$  is an arbitrary parameter which parameterizes the squashing of the  $S^3$  which is the conformal boundary.

The AdS TN and AdS TB have the same asymptotic behavior for  $k = 1$ . For  $|k| > 1$ , if we identify  $|k|$  points on the  $S^1$  fiber of the AdS TN solution, we obtain a space which has the same boundary structure as the AdS TB solution with parameter  $k$ .

This identified AdS TN solution, however, has a conical singularity at the origin, which can be smoothed out.

For computation for the analogue of the ADM mass, and action for these solutions, we need to compare it to some reference metric which has the same boundary behavior. The reference metric is taken to be the AdS Taub–NUT metrics discussed above, with appropriate identifications along the  $S^1$  fiber to get the same asymptotic structure. Then, the Hamiltonian calculation reveals that there are two AdS Taub–Bolt metrics with the same temperature, but different masses. The one with the lower masses is thermodynamically unstable, since it has a negative specific heat. In addition, as in the AdS case, there is a phase transition in the system (for  $k = 1$ ). The AdS Taub–NUT solution exists for all temperatures. However, the AdS Taub–Bolt solution can only exist for temperatures above a minimum value  $T_0$ .

Furthermore, since we have multiple bulk solutions with the same boundary behavior, the partition function of the boundary CFT will receive contributions from the different bulk spaces with the same boundary behavior. To determine which one dominates, we need to evaluate the action for these solutions. It can be shown [34] that for temperatures below  $T_1$  ( $> T_0$ ), the AdS Taub–NUT background is favored, whereas for temperatures above  $T_1$ , the AdS Taub–Bolt solution dominates, and the AdS Taub–NUT background will decay into it. This presumably corresponds to a confinement/deconfinement phase transition for the boundary theory living on the squashed  $S^3$ .

### 3 Static Lorentzian space-times, $\Lambda < 0$

In the previous section, we discussed Euclidean situations, where specifying the boundary values of the various fields at the conformal boundary determines the bulk configurations, in some cases uniquely, and in others up to a few discrete choices. The situation in Lorentzian signature is more subtle. The normalizable mode solutions to the equations of motion, which exist in Lorentzian signature, can be arbitrarily added to a bulk solution with a given boundary behavior without affecting the boundary behavior. The choice of the normalizable part of the solution corresponds to the choice of state in the conformal field theory in which the partition function and hence the correlation functions are computed. Here again we will only deal with the case of four dimensional bulk, as it is one of the most studied cases [4].

#### 3.1 $\partial M = \mathbb{R} \times S^2$ , $M = \mathbb{R} \times \mathbb{R}^3 = \mathbb{R}^4$

This is the usual Lorentzian AdS/CFT setup. The boundary CFT lives on  $\mathbb{R} \times S^2$ . Given a certain boundary metric (with non-negative Ricci scalar), a bulk metric always exists with that boundary behavior [4]. The uniqueness of such a metric is not guaranteed in general. However for the boundary metric  $ds^2 = -dt^2 + ds_{S^2}^2$ , there is a unique

globally static bulk metric with conformal compactifiable smooth acausal equal time slices. This is just the standard metric of  $\text{AdS}_4$  [4].

### 3.2 $\partial M = \mathbb{R} \times S^2$ , $M = \mathbb{R} \times (\mathbb{R}^+ \times S^2)$

Looking now at the same boundary manifold but at  $M = \mathbb{R} \times (\mathbb{R}^+ \times S^2)$  as a bulk manifold, we find a rather different situation. Taking the boundary metric to be again  $ds^2 = -dt^2 + ds_{S^2}^2$ , one can check that the following family of 1-parameter bulk metrics all have the required asymptotics – these are the Lorentzian AdS Schwarzschild black holes with the metric

$$ds^2 = -U^2 dt^2 + U^{-2} dr^2 + r^2(d\theta^2 + \cos^2 \theta d\phi^2), \quad (9)$$

where

$$U^2 = 1 + r^2 - \frac{2m}{r}, \quad m > 0. \quad (10)$$

This background corresponds to a thermal state in the boundary CFT. It differs from the usual setup in (3.1) by the choice of the state on the boundary.

One can also show that these AdS Schwarzschild black hole metrics are the unique globally static metrics smooth up to the horizon with conformal compactifiable smooth acausal equal time slices [4].

The dual CFT corresponding to the boundary conditions of both the global  $\text{AdS}_4$  metric described in Section 3.1 and these AdS Schwarzschild black holes is a CFT defined on a spatial manifold  $S^2$ . To discuss this theory at finite temperature, one effectively needs to calculate the partition function on  $S^2 \times S^1$ , where the radius of the extra  $S^1$  factor is related to the inverse of the temperature, and it can be thought of as the time direction, Wick rotated to Euclidean signature. The calculation of the partition function then follows the one we had in Section 2, for Euclidean space-times. Therefore the Hawking–Page phase transition occurs here and is seen in the field theory as a confinement–deconfinement transition. Of course in cases where the field theory is not in a finite temperature, it is hard to tell which geometry would dominate the partition function.

### 3.3 $\partial M = \mathbb{R} \times T^2$ , $M = \mathbb{R} \times (D^2 \times S^1)$

Let us look at the boundary metric  $ds^2 = -dt^2 + ds_{T^2}^2$ . Then one can show [4], [26] that all the globally static metrics on  $M$  with such asymptotics and with conformal compactifiable smooth acausal equal time slices are of the “AdS soliton” type discussed by Horowitz and Myers [38]:

$$ds^2 = -r^2 dt^2 + U^{-2} dr^2 + U^2 d\phi^2 + r^2 d\theta^2, \quad (11)$$



where

$$U^2 = r^2 - \frac{2m}{r}, \quad m > 0. \quad (12)$$

$\phi$  is a periodic angle of period  $\beta = \frac{4\pi}{3(2m)^{1/3}}$ , and  $\theta$  is of arbitrary period.

One can also show that for any given boundary metric on  $T^2$  there are countably many such filling metrics parameterized by the choice of an  $S^1 = \partial D^2$ .

These geometries have the interesting property that their mass is negative (relative to the choice where conformal flatness means zero energy). This fact has a natural interpretation on the CFT side – it was shown in [38] that the corresponding CFT has a negative Casimir energy, related to the breaking of supersymmetry on the CFT by the boundary conditions on the fermions. In fact it was conjectured in [38] that the “AdS soliton” metrics are the lowest energy solutions with these given boundary conditions.

### 3.4 $\partial M = \mathbb{R} \times T^2, M = \mathbb{R} \times (\mathbb{R}^+ \times T^2)$

In this case, taking again the boundary metric to be  $ds^2 = -dt^2 + ds_{T^2}^2$ , one can show that there is a 1-parameter family, this time of toroidal black holes with these asymptotics. This family of toroidal Kottler metrics is given by

$$ds^2 = -U^2 dt^2 + U^{-2} dr^2 + r^2 d\phi^2 + r^2 d\theta^2, \quad (13)$$

where as before

$$U^2 = r^2 - \frac{2m}{r}, \quad m > 0, \quad (14)$$

and where both  $\phi, \theta$  are periodic of arbitrary period. These metrics have a horizon at  $r^4 = 2m$  which is  $\mathbb{R} \times T^2$ . As before these filling metrics are the unique ones which are globally static and with conformal compactifiable smooth acausal equal time slices [4].

The energy of these black holes is greater than that of the AdS solitons of the same boundary structure, in accordance with the conjecture made by Horowitz and Myers [38]. However, a thermodynamical analysis [55], [16], [57] shows that the free energy of these black holes can be greater or smaller than that of the solitons, leading to a phase transition, somewhat similar to the Hawking–Page transition we mentioned earlier. It has been shown that small, hot black holes are unstable and decay to small, hot solitons. Large cold black holes are stable. The order parameter for the transition depends both on the horizon area and on the temperature of the black hole (which are two independent parameters for these black holes). On the side of the CFT, this phase transition can be related to a confinement–deconfinement transition [55], [52].

## 4 Multiple boundary configurations

In cases where the boundary of space-time has multiple disconnected components, the issue of a dual holographic description is more involved. On the one hand the holographic theory is defined on a union of disjoint manifolds. There is no obvious way in which the theories on the different manifolds are coupled, and apriori it seems natural to expect that the holographic theory would just be the product of the theories on each one of the boundary components. On the other hand the bulk theory seems to induce correlations between the different boundary regions.

This seeming puzzle bears a somewhat different nature depending on whether one is discussing Euclidean or Lorentzian settings.

In the Lorentzian case, for asymptotically AdS space-times (i.e.  $\Lambda < 0$ ), a *topological censorship theorem* was proved [25], which basically states that under certain conditions, the presence of multiple boundaries forces the bulk to be separated by horizons, in such a way that different boundary components are not causally connected through the bulk. This implies that the different holographic theories living on the different boundary components would indeed be uncorrelated and will not interact dynamically. The only correlations could be ones in initial states of the theory.<sup>2</sup> Let us state the topological censorship theorem more precisely now: Let  $\mathcal{M}'$  be a globally hyperbolic space-time with boundary, with timelike boundary  $\mathcal{I}$  that satisfies the average null energy condition.<sup>3</sup> Let  $\mathcal{I}_0$  be a connected component of  $\mathcal{I}$  of  $\mathcal{M}'$ . Furthermore assume that either (i)  $\mathcal{I}_0$  admits a compact spacelike cut or (ii)  $\mathcal{M}'$  satisfies the generic condition.<sup>4</sup> Then  $\mathcal{I}_0$  cannot communicate with any other component of  $\mathcal{I}$ , i.e.  $J^+(\mathcal{I}_0) \cap (\mathcal{I} \setminus \mathcal{I}_0) = \emptyset$ .

In the Euclidean case, for asymptotically AdS Einstein space-times, the puzzle is avoided due to a theorem by Witten and Yau [62], basically stating that if one of the boundary components has  $R > 0$ , then the boundary is connected. This theorem was later generalized by Cai and Galloway [17] to cases where the boundary has zero scalar curvature. Let us state the general theorem: Let  $M^{n+1}$  be a complete Riemannian manifold which admits a conformal compactification, with conformal boundary  $N^n$ , and with the Ricci tensor of  $M$  satisfying  $\text{Ric} \geq -ng$  such that  $\text{Ric} \rightarrow -ng$  sufficiently fast on approach to conformal infinity.<sup>5</sup> If  $N$  has a component of nonnegative curvature, then the following holds: (i)  $N$  is connected; (ii) If  $M$  is

---

<sup>2</sup>One example for this is the case of Schwarzschild AdS black holes, and in particular the BTZ black hole. These were studied in [48] and we would make a few comments about them below.

<sup>3</sup>The average null energy condition states that for each point  $p$  in  $\mathcal{M}$  near  $\mathcal{I}$  and any future complete null geodesic  $s \rightarrow \eta(s)$  in  $\mathcal{M}$  starting at  $p$  with tangent  $X$ ,  $\int_0^\infty \text{Ric}(X, X) ds \geq 0$  ( $\text{Ric}(X, X)$  denotes  $R_{ab}X^a X^b$ ). This condition is satisfied by space-times created from physically reasonable matter sources.

<sup>4</sup>A space-time satisfies the generic condition if every timelike or null geodesic with tangent vector  $X$  contains a point at which  $X^a X^b X_{[c} R_{d]ab[e} X_{f]}$  is nonzero.

<sup>5</sup>I.e.  $r^{-2}(\text{Ric} + ng) \rightarrow 0$  as  $r \rightarrow 0$  where the bulk metric is expanded in a neighborhood of the boundary as  $g = \frac{1}{r^2}(dr^2 + g_r)$ , and the conformal boundary is at  $r \rightarrow 0$ . For some discussion on a physical interpretation of these conditions, see [51].

orientable, then  $H_n(\bar{M}, Z) = 0$ ; (iii) The map  $i_*: \Pi_1(N) \rightarrow \Pi_1(\bar{M})$  ( $i$  = inclusion) is onto.

This theorem therefore implies that the puzzle we described does not arise in asymptotically AdS Einstein space-times with nonnegative boundary curvature. One might wonder then about the case where the boundary has negative curvature. In such cases, it can be shown that the holographic theory living on the negative curvature boundary would be unstable for any boundary dimension  $n \geq 3$ . However, the Witten–Yau theorem can be avoided if we turn on extra supergravity fields, and thus look not at Einstein space-times, but rather at space-times obeying the more general supergravity equations. The instability related to negative curvature boundary can also be avoided if we look at specific settings of 3-dimensional bulk (i.e.  $n = 2$ ). Such examples will be presented in Section 4.3.

Let us now discuss a few examples of multi boundary situations where one can say something about the AdS/CFT correspondence. We shall focus in the case where the bulk is 3-dimensional, where things are better known. In fact in the case of three dimensions, the only solution to Einstein’s equations with a negative cosmological constant is locally  $\text{AdS}_3$ . Different space-times can only differ from each other by global identifications.

Starting from Lorentzian configurations, one can build configurations with multiple boundary components by taking a 2-dimensional slice of global  $\text{AdS}_3$ , cutting and gluing it along geodesics and then letting it evolve in time [2]. Such constructions lead, in agreement with the topological censorship theorem, to space-times with the same number  $h$  of boundaries and horizons, and with any number  $g$  of handles behind the horizons.

In the special case where there are only two boundaries ( $h = 2, g = 0$ ), the space-time describes the eternal BTZ black hole.

## 4.1 Eternal BTZ

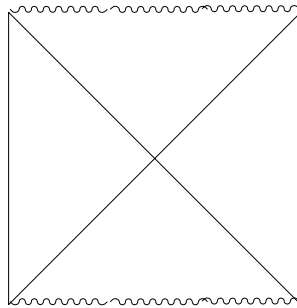


Figure 1. Penrose diagram of the eternal BTZ

An eternal BTZ black-hole has two boundaries. More precisely, it has two asymptotically  $\text{AdS}_3$  regions each of which is separated from the other by a horizon (see the Penrose diagram in Fig. 1). Locally, it is isometric to  $\text{AdS}_3$  but differs from it in its global properties. Three dimensional Anti-de Sitter space is a maximally symmetric space of constant negative curvature. It is the hyperboloid

$$\begin{aligned} \text{AdS}_3 &\hookrightarrow \mathbb{R}^{2,2} \\ -u^2 - v^2 + x^2 + y^2 &= -l^2, \end{aligned} \quad (15)$$

in flat  $\mathbb{R}^{2,2}$ . By construction, the isometry group is  $\text{SO}(2, 2)$ . The Killing vectors of the metric generate the Lie algebra  $\mathfrak{so}(2, 2)$  of the isometry group, and are described in terms of the embedding space  $\mathbb{R}^{2,2}$  as

$$J_{ab} = x_b \partial_a - x_a \partial_b, \quad (16)$$

with  $x^a \equiv (u, v, x, y)$  and  $x_a = \eta_{ab} x^b$ , with  $\eta_{ab} = (-, -, +, +)$ .

The BTZ black holes are obtained by identifying  $\text{AdS}_3$  by the discrete action generated by the Killing vector [10]

$$\xi_{\text{BTZ}} = \frac{r_+}{l} J_{12} - \frac{r_-}{l} J_{03} - J_{13} + J_{23}. \quad (17)$$

In the non-extremal case,  $r_+^2 - r_-^2 > 0$  and by a  $\text{SO}(2, 2)$  transformation,  $\xi_{\text{BTZ}}$  can be brought into the form:

$$\xi'_{\text{BTZ}} = \frac{r_+}{l} J_{12} - \frac{r_-}{l} J_{03}. \quad (18)$$

The mass and angular momentum of the black hole are given by

$$M = \frac{1}{l^2} (r_+^2 + r_-^2), \quad J = \frac{2}{l} r_+ r_-. \quad (19)$$

The extremal black hole is obtained by taking the limit  $r_+ \rightarrow r_-$  in (17), so that the generator becomes

$$\xi_{\text{BTZ}} \rightarrow \frac{r_+}{l} (J_{12} - J_{03}) - J_{13} + J_{23}. \quad (20)$$

The holographic description of the space-time is given in terms of two non-interacting identical CFTs. In Lorentzian AdS/CFT, a holographic description includes the specification of a state in the CFT. The relevant state in two CFTs which describes the BTZ background is a particular entangled state given by

$$|\Psi\rangle = \sum_n e^{-\frac{\beta}{2} E_n} |E_n\rangle_1 \times |E_n\rangle_2, \quad (21)$$

where  $|E_n\rangle_{1,2}$  denotes an energy eigenstate in the two CFTs and  $\beta = \frac{\pi l}{r_+}$ .

Computation of the correlation functions involving only operators in one of the two CFTs in this entangled state lead to thermal correlation functions:

$$\langle \Psi | \mathcal{O}_1 | \Psi \rangle = \sum_n e^{-\beta E_n} \langle E_n | \mathcal{O}_1 | E_n \rangle_1 = \text{Tr}(\rho_\beta \mathcal{O}_1), \quad (22)$$

where  $\rho_\beta$  is the thermal density matrix.

We will come back to the eternal BTZ in Section 5.

## 4.2 Two null cylinder boundaries

This space has two boundaries, each of which are null cylinders, *i.e.* flat space with a compact null direction. The metric is given by

$$ds^2 = l^2 \left( -(dt)^2 + (d\phi)^2 + 2 \sinh(2z) dt d\phi + (dz)^2 \right), \quad (23)$$

with  $\phi$  being an angular coordinate taking values in  $[0, 2\pi)$ , and the coordinates  $(z, \phi, t)$  give a global parameterization of the space. The two boundaries are at  $z \rightarrow \infty$ . Unlike the BTZ, the boundaries are not separated by a horizon. The space is stationary but not static. It is a quotient of  $\text{AdS}_3$  by action of a subgroup of  $\text{SO}(2, 2)$  isomorphic to  $\mathbb{Z}$ .

$$P \rightarrow e^{t\xi} P, \quad t = 0, \pm 2\pi, \pm 4\pi, \dots \quad \text{for all } P \in \text{AdS}_3 \quad (24)$$

where

$$\xi = \frac{1}{2} (J_{02} + J_{13}). \quad (25)$$

This generator is a linear combination of a boost in the  $ux$ -plane and  $vy$ -plane in the embedding space  $\mathbb{R}^{2,2}$ .

The metric (23) is an  $S^1$  fibration over  $\text{AdS}_2$ . Indeed, we can rewrite this metric as

$$g = l^2 \left( -\cosh^2(2z) dt^2 + dz^2 + (d\phi + \sinh(2z) dt)^2 \right). \quad (26)$$

Compactifying on  $\phi$  now gives the metric

$$\begin{aligned} g_2 &= -\cosh^2 2z dt^2 + dz^2, \\ A_1 &= \sinh 2z dt. \end{aligned}$$

The metric is precisely that of  $\text{AdS}_2$ , but there is also a constant electric field.<sup>6</sup>

The fact that the boundaries are null cylinders implies that the boundary theory is defined through a discrete light cone quantization procedure. The exact map between the bulk and the boundary theory is still somewhat mysterious. For more details, see [8].

---

<sup>6</sup>The field is constant in the sense that the field strength of the  $U(1)$  connection is proportional to the  $\text{AdS}_2$  volume form.

### 4.3 Wormholes

Now looking at Euclidean setups, in order to describe multiple boundary spaces (i.e. wormholes), we must look at configurations which do not satisfy the conditions of the Witten–Yau theorem. One possibility is to look at spaces which have two boundaries, each being a Riemann surface of genus  $g \geq 2$ ,  $\Sigma_g$  [49]:

$$ds^2 = d\rho^2 + \cosh^2 \rho ds_{\Sigma_g}^2. \quad (27)$$

The boundaries have constant negative curvature, so the Witten–Yau theorem does not apply here, but unlike in higher dimensions, the two-dimensional field theories on these Riemann surfaces are well-defined and stable. Such spaces can be created by a quotient of  $H_3$  by a discrete subgroup  $\Gamma \in \text{SL}(2, \mathbb{C})$ . The two Riemann surfaces have the same genus, but may differ in their moduli  $t^\alpha$ . Performing a quotient of  $\text{SL}(2, \mathbb{C})$  by a Fuchsian subgroup  $\Gamma$  results in two Riemann surfaces which have the same moduli. Performing a quotient by a quasi-Fuchsian subgroup results in two Riemann surfaces of different moduli. In fact, according to the Bers simultaneous uniformization theorem [13] the quasi-fuchsian space of a Riemann surface  $\Sigma_{g,n}$  of genus  $g$  and  $n$  punctures:  $\mathcal{QF}(\Sigma_{g,n})$  is homeomorphic to pairs of points in the Teichmüller space of  $\Sigma_{g,n}$ :  $\text{Teich}(\Sigma_{g,n}) \times \overline{\text{Teich}(\Sigma_{g,n})}$ .

In such cases, the puzzle we described previously for multiple boundary spaces is apparent, and it is not clear whether correlations would or would not exist between the two boundary holographic theories [49]. Actually it is interesting to note that it is also possible to construct by discrete quotients of  $H_3$ , spaces with a single boundary which could be any Riemann surface  $\Sigma_{n,g}$  (this is guaranteed by the classical retrosection theorem). In this case the discrete subgroup is a Schottky group, and such constructions were described in [40]. It would be interesting to see if the geometry with disconnected boundaries (i.e. a union of two such spaces, each with a single Riemann surface as a boundary) has larger or smaller action than the geometry that connects the two boundaries. If the disconnected geometry is the dominant one, then it is possible that correlations between the two boundaries will indeed be suppressed.

Another way to avoid the Witten–Yau theorem and still build a Euclidean space-time with multiple boundaries of positive curvature, is to add to the pure Einstein gravity some gauge fields. An example in 4 dimensions was built in [49], where an  $\text{SU}(2)$  gauge field was introduced, and the 4-dimensional action is

$$S \sim \int d^4x \sqrt{g} [-R + \Lambda + F_{\mu\nu}^a F^{a\mu\nu}]. \quad (28)$$

Here the field strength is given in terms of the gauge field by  $F = dA + A^2$  and the cosmological constant is normalized to be  $\Lambda = -6$ .<sup>7</sup> The following two-boundary

---

<sup>7</sup>It was shown in [49] that this action is a consistent reduction of 11-dimensional supergravity.

solution for this action was constructed:

$$\begin{aligned} ds^2 &= d\rho^2 + e^{2w} ds_{S^3}^2; \quad e^{2w} = \frac{1}{2}(\sqrt{5} \cosh 2\rho - 1) \\ A &= \frac{1}{2}\omega^a, \end{aligned} \tag{29}$$

where  $\omega^a$  are the left-invariant,  $\mathfrak{su}(2)$ -valued one-forms on  $S^3$ , such that  $ds_{S^3}^2 = \frac{1}{4}\omega^a\omega^a$ . For large  $\rho$  the geometry becomes that of  $H_4$  and the boundaries are 3-spheres, and for  $\rho = 0$  we have a finite throat size. The precise way in which the AdS/CFT correspondence would work here is quite mysterious. It would be very interesting to understand its exact meaning and interpretation for such configurations. For more details, other examples of wormhole setups, and some speculations regarding possible resolutions of the AdS/CFT puzzle for such configurations see [49].

## 5 Some interesting questions

### 5.1 The information paradox

Hawking showed that black holes are not really black but in fact emit a thermal radiation. This follows from a semi-classical analysis. We now imagine matter in a pure state falling into the black hole. Hawking's semi-classical analysis reveals that it will eventually be radiated out as thermal radiation, which is a mixed state. This exposes a paradox: in quantum mechanics, pure states evolve to pure states. How then can a pure state evolve into a mixed state in a black hole background. The information about the initial pure state seems to have been lost inside the black hole. This has been called the information loss paradox [33] – after matter falls into the black hole, the correlators with infalling matter decay exponentially, so if we wait a long enough time, the correlation functions will eventually vanish. This violates unitarity. In AdS, this poses a particularly sharp paradox since asymptotically AdS black holes can live for ever.

In Section 4.1 we described the eternal BTZ black hole and its description in the two boundary CFTs in terms of an entangled state. In [48], Maldacena considered a deformation of the thermal Schwarzschild AdS state by adding an operator to the second boundary and showed that the correlations indeed die off exponentially. This change in the black hole state, although minor, is still detectable: the one point function of the same operator in the first CFT which was previously zero is now non-zero, but dies exponentially fast at a rate  $e^{-\frac{ct}{\beta}}$  where  $c$  is a numerical constant. The puzzle is that correlation functions of the boundary CFT cannot decay at late times since it signals a loss of unitarity (one can in fact show that to be consistent with unitarity, the correlations could be as small as  $e^{-cS}$  where  $S$  is the entropy of the ensemble and  $c$  is a numerical factor). The resolution comes from the fact that in AdS/CFT,

we need to sum over all geometries with a prescribed boundary behavior. In fact, there are other geometries than just the Schwarzschild geometry that we have so far considered. The additional geometry that provides the relevant effect consists of two separate global AdS spaces with a gas of particles on them in an entangled state. This geometry contributes with a small weight because it has a very small free energy compared to the Schwarzschild geometry, but it indeed gives a non-decaying answer of order  $e^{-cS}$  as expected from constraints of unitarity; see also [11], [37], [15], [54] and [39] for further discussion. In the last paper it is shown that although the sum over geometries does yield a non-decaying answer, it seems extremely difficult to obtain the required quasi-periodic answer for correlation functions this way, unless one manages to perform the sum over geometries in closed form.

There is another picture of black holes due to Mathur et al that has got some attention recently which we describe in more detail in Section 5.3. According to this picture, the black hole should be thought of as an ensemble of classical geometries, each of which has no horizon. The absence of a horizon in each of the different geometries evades the information loss problem.

## 5.2 Singularity beyond the horizon

In the AdS/CFT correspondence, the region outside the horizon of a black hole is represented holographically by a boundary CFT at finite temperature. Since the black hole singularity is inside the horizon, at first sight it seems that AdS/CFT cannot be used to gain insight into the nature of the singularity. However, the situation is more subtle for the eternal black holes, which have multiple asymptotic regions. For example, the eternal BTZ black hole has two asymptotically AdS regions. The holographic dual is given by two decoupled CFTs living on the two boundaries, living in an entangled state as described in Section 4.1. For correlations functions where all the operators belong to one of the two CFTs, we can trace over the states of the other CFT leading to correlation functions in a thermal state. Such correlation functions will not contain any non-trivial information about the physics beyond the event horizon. However, correlation functions of operators in each of the two CFTs will contain information about the region beyond the horizon. This is most easily seen by using the geodesic approximation to compute the correlators. For example, for a two point function of operators inserted on the two boundaries, the WKB approximation is good for bulk fields of large mass. In this approximation, space-like geodesics dominate the contribution to the 2-point function, and these geodesics traverse the region behind the horizon. In the case of the BTZ black hole, these two point computations can be carried out exactly, by using appropriate bulk to boundary propagators, and then moving the bulk point to the boundary and removing an overall rescaling. In [41], Kraus et al defined these amplitudes by an analytic continuation procedure from Euclidean signature. This analytic continuation can be done in different ways. In one way of performing the analytic continuation (I), in Lorentzian signature, the contribution



comes from only the region behind the horizon whereas in the other procedure (II), the contribution comes from both the region outside and behind the horizon. Since the two analytic continuation procedures are equivalent and finite, (I) manifestly so, while in (II), the singularity can be regulated by an  $i\epsilon$  prescription inherent in the analytic continuation procedure, and the contribution from the past and future singularity can be shown to cancel. Its not entirely clear how much information behind the horizon can really be inferred from this procedure: the fact that we can obtain the same correlation function by integrating in the region outside the horizon seems to suggest that no real information behind the horizon can really be contained in these correlations functions. Similar analysis can be carried out for rotating BTZs [43], [7].

The situation for AdS Schwarzschild black holes in higher dimensions is more involved. As was shown in [23], the Penrose diagram is not a square, which results in a contribution from an almost null geodesic in real coordinates which bounces off the singularity at a fixed boundary time  $t_c$ . This would imply that in the CFT correlation function, there is a light cone singularity at  $t = t_c$ . This is problematic because such a singularity is ruled out in the CFT on very generic grounds. However, it was shown in [23] that the CFT correlation function is in fact dominated by a complexified geodesic. There is a branch cut in the CFT correlation function at  $t = t_c$ , and the information about the black hole singularity is contained in the analytic structure near  $t = t_c$ .

### 5.3 Where are the microstates of the black hole?

An important breakthrough in our understanding of black holes was the realization that the horizon area of black holes has all the properties of thermodynamical entropy [12]. This seemed to suggest that there exists a large number of microstates (of the order of the exponential of the horizon area in appropriate units) building up the black hole. In some settings, this large number of states can be reproduced from the dual holographic field theory. However, the question remains how all these different states are manifested in terms of the actual gravity description of the black hole.

In a recent series of papers [46], it was suggested that in fact the black hole is not one classical solution having a singularity and a horizon, but rather is a “coarse grained” description of an ensemble of different geometries, each being completely regular, and each corresponding to a microstate in the dual field theory. These geometries are all very similar to each other and to the ‘naive’ black hole geometry, when probed with particles of large wavelength, but differ in a small region which defines the location of a ‘horizon’. One must note that this horizon has nothing to do with the classical horizon we are used to. it is not a special surface and there is no singularity inside it. It is just the characteristic location where all the different geometries start to differ from each other.

Such geometries were actually built for a specific system of branes – the two charge rotating D1–D5 system (characterized by the charges of the branes:  $Q_1$  and  $Q_5$  respectively and by an angular momentum related parameter  $a$ ), which is supersymmetric

(1/4 BPS) [45]. At some scaling limit, it describes  $\text{AdS}_3 \times S^3$ , whose holographic dual is a 1+1 dimensional CFT with  $\text{SO}(4)$  symmetry. For this system the geometries built are asymptotically flat (at  $r \gg (Q_1 Q_5)^{1/4}$ ), and have a finite throat, which at some length scales ( $(Q_1 Q_5)^{1/4} \gg r \gg a$ ) describes an  $\text{AdS}_3 \times S^3$  geometry, and deeper ( $a > r$ ) describes a different geometry, whose details depend on the specific state associated. These systems are also related by U-duality to the 1/4 BPS supertubes [50] and for critical values of the angular momentum have a similar ‘blow-up’ mechanism [45].

One problematic aspect of these systems is that their macroscopic entropy really vanishes, as the number of microscopic states, although finite, is too small to give any macroscopic entropy. It would be nice if these ideas could be also shown for the 3-charge D1–D5-momentum system (which is 1/8 BPS), where the macroscopic entropy is nonzero. The main problem is that it is not known how to build a general family of geometries dual to all these microstates. First attempt in this direction have recently been made in [44], [29].

Another approach to counting the geometrical microstates of this system is to try and manipulate the expression for the elliptic genus of the conformal field theory and rewrite it as a sum over different geometries with  $\text{AdS}_3 \times S^3$  asymptotics [19], [22].

## 6 Chern–Simons theory

Pure Gravity in three dimensions has many special and interesting features, making it, on the one hand a convenient laboratory for studying gravity and holography, but on the other hand less generic and harder to generalize to a different number of dimensions. Many of these features are related to the fact that three dimensional gravity can be rewritten as a topological Chern–Simons theory [58].<sup>8</sup>

Starting with the regular Einstein action

$$S = \frac{1}{16\pi G} \int_M d^3x \sqrt{|g|} [R - 2\Lambda], \quad (30)$$

one can change variables from the metric  $g_{\mu\nu}$  to the first order forms – the dreibeins  $e_\mu^a$  (such that  $g_{\mu\nu} = e_\mu^a e_\nu^a$ ) and the spin connections  $\omega_\mu^a \equiv \frac{1}{2} \epsilon^a_{bc} \omega_\mu^{bc}$ , where the action is re-written as

$$S = \frac{2}{16\pi G} \int_M d^3x \left[ e^a \wedge \left( d\omega^a + \frac{1}{2} \epsilon_{abc} \omega^b \wedge \omega^c \right) + \frac{\Lambda}{6} e^a \wedge e^b \wedge e^c \right]. \quad (31)$$

---

<sup>8</sup>This is also true for three dimensional supergravity [36].

Then changing variables from  $e^a, \omega^a$  to  $\mathcal{A}_{L,R}^a = \omega^a \pm \frac{1}{\sqrt{-\Lambda}} e^a$ , the action becomes

$$S = k_L S_{\text{CS}}[\mathcal{A}_L] - k_R S_{\text{CS}}[\mathcal{A}_R],$$

$$S_{\text{CS}}[\mathcal{A}] \equiv \frac{1}{2} \int_M \text{Tr}(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}), \quad (32)$$

and  $k_L = \frac{\sqrt{-\Lambda}}{8\pi G}$ ,  $k_R = k_L^*$ . In (32) we regard  $\mathcal{A}_{L,R}$  as taking value in a Lie algebra. In case we are discussing Lorentzian gravity they are two independent 1-forms, each taking values in  $\mathfrak{sl}(2, R)$ , and in case we are discussing Euclidean gravity, they are complex conjugates of each other, taking values in  $\mathfrak{sl}(2, \mathbb{C})$ .

The fact that one can recast pure gravity as a gauge theory which is topological is very appealing. However, the change of variables we presented here between the second and first order formalisms is naive and ignores an important subtlety. Namely, the mapping is not one-to-one, and it is not clear whether one should include also degenerate  $\mathcal{A}_{L,R}$  or not. Therefore there might be problems defining the measure in the path integrals for the gravity action and for the Chern–Simons theory. Another possibly related problem is that it seems the Chern–Simons theory cannot account for the black hole entropy (as calculated from its horizon area), and predicts a much smaller number of states [42]. One therefore might question whether the theories actually exist. There are also other interesting subtleties related to the AdS/CFT correspondence in this setting, which we would not get into here (see for example [32]).

One interesting aside is that in fact many 3-manifolds admit a hyperbolic structure (i.e. admit metrics of constant negative curvature). For such compact manifolds, there exists a natural complexification of the volume of the manifold, which involves the Chern–Simons topological invariant, and has good analytic properties [56]:

$$Z(M) \sim \exp \left[ \frac{2}{\pi} \text{Vol}(M) + 4\pi i \text{CS}(M) \right], \quad (33)$$

(where above we set  $\Lambda = -1$ ). It is not clear if and what would be the importance and interpretation of this invariant in the context of the AdS/CFT correspondence. For some discussion of this invariant and its possible applications see [31].

## 7 Time dependence

There are many interesting issues that arise in time dependent or cosmological space-times. For example, there are generically multiple natural vacua that we can choose in such backgrounds. Also, most such backgrounds exhibit cosmological particle production. The holographic theory reflects these phenomena in an interesting way. We now discuss an explicit example of such a background.

## 7.1 AdS bubbles of nothing

By starting with the AdS Schwarzschild solution and performing a double analytic continuation, we obtain interesting time dependent backgrounds which are called AdS bubbles of nothing [14], [9], [27]. The AdS Schwarzschild metric is given in (9). The analytic continuation  $t \rightarrow i\chi$  and  $\theta \rightarrow i\tau$  yields a time dependent space-time which is a vacuum solution to five dimensional gravity with a negative cosmological constant:

$$ds^2 = \left(1 + \frac{r^2}{l^2} - \frac{2m}{r}\right) d\chi^2 + \left(1 + \frac{r^2}{l^2} - \frac{2m}{r}\right)^{-1} dr^2 + r^2(-d\tau^2 + \cosh^2 \tau d\phi^2). \quad (34)$$

This is a smooth space-time if  $\chi$  is periodic with period  $\frac{4\pi r_+ l^2}{3r_+^2 + l^2}$ . Here  $r_+$  is the minimum value of the coordinate  $r$  and is the largest positive root of the equation  $r_+^3 + l^2 r_+ - 2ml^2 = 0$ . For a fixed  $\tau$  and for  $r > r_+$ , the space is basically  $S_\chi^1 \times S_\phi^1$ . As  $r \rightarrow r_+$ , the circle parameterized by  $\chi$  collapses and the circle parameterized by  $\phi$  approaches a finite size  $r_+^2 \cosh^2 \tau$ . This circle is the boundary of a bubble of nothing. The metric on the boundary of the bubble is 2d de Sitter space. This space is asymptotically AdS with the conformal boundary being two dimensional de Sitter space-times a circle. So the holographic dual to this bubble lives on  $dS_2 \times S^1$  [9].

Similar time dependent space-times can be constructed by performing double analytic continuations of AdS–Kerr and Reissner–Nordstrom AdS black holes.

## 8 Zero and positive cosmological constant

In the previous sections solutions of the Einstein equations with a negative cosmological constant and in particular Anti-de Sitter space played a prominent role. It is an obvious question to what extent similar results can be obtained for spaces with zero or a positive cosmological constant. Much less is known in these cases, and it is in fact not clear to what extent a meaningful holographic duality can be formulated.

### 8.1 Positive cosmological constant

The maximally symmetric solution of the Einstein equations with a positive cosmological constant is de Sitter space. In Euclidean signature it is simply a  $d$ -sphere, whereas in Lorentzian signature it is the time-dependent geometry

$$ds^2 = -dt^2 + \cosh^2 t d\Omega_{d-1}^2, \quad (1)$$

with  $d\Omega_{d-1}^2$  the metric on a round  $(d-1)$ -sphere. The metric (1) describes a sphere that contracts exponentially in the past until it reaches a fixed size, and then expands

exponentially again. According to recent experimental data, the present day expanding universe is well described by de Sitter space.

Whether or not quantum gravity (or rather, string theory) on a space like (1) is dual to a field theory of some sort is unclear. Attempts to find such field theories run into various kinds of problems (see e.g. [61]). Unfortunately, despite a lot of recent work, a clean explicit example of a solution to the string theory equations of motion of the form (1) is still lacking. Such a solution would obviously be very helpful in exploring the physics of quantum gravity in de Sitter space.

The space (1) has two boundaries and a cosmological horizon. Associated to the cosmological horizon is a finite Hawking temperature, and in addition it has a finite area, similar to what one has for a black hole horizon. This suggests that there might exist some version of holography which applies to cosmological horizons and associates a finite entropy to them. If one additionally believes that a version of black hole complementarity applies to cosmological horizons – i.e. the Hilbert space of a single observer is sufficient to describe both sides of the horizon – then a dual description of de Sitter space might involve a theory with a finite dimensional Hilbert space. In such a theory there is not enough resolving power to measure arbitrary small distances, and therefore it can at best yield a dual of (1) which is a good description for a finite amount of time but not asymptotically as  $t \rightarrow \pm\infty$ .

If we try to apply the AdS/CFT philosophy more directly to (1), we should first find all possible solution of the Einstein equations that are asymptotically identical to (1). It is known [24] that for sufficiently small deformations of the boundary metrics a smooth solution with the same asymptotic behavior still exists. However, the situation for large deformations has not been resolved. Under significantly large perturbations de Sitter space can break into pieces and in particular disconnected geometries (for example so-called big bang/crunch geometries) will start to contribute [28], [6]. Therefore it is quite possible that the sum over geometries involved in a putative dS/CFT correspondence will involve a much larger set of metrics and geometries than just (1).

Perhaps a better strategy is to first study some simpler aspects of de Sitter space before engaging in a full-fledged holographic correspondence. For example, whether or not a positive mass theorem for de Sitter space exists and if so what its precise formulation is, is still an open problem. A preliminary positive mass theorem was described in [6], and it was verified in many different examples, but a closer look [18] suggests that its formulation is not quite complete as it stands.

## 8.2 Zero cosmological constant

The case of zero cosmological constant is at least as problematic as the case of a positive cosmological constant. The maximally symmetric solution of the Einstein equations is Minkowski space (or smooth quotients thereof). There have been a few attempts at finding a dual description of quantum gravity in Minkowski space. First, one can try to find a dual description of a subset of Minkowski space by putting suitable

“holographic screens” in it, see [21]. This has not led to a concrete dual description however. Another approach involves taking a decompactification limit of AdS/CFT [53]. This turns out to be quite difficult and has not led to a precise dual description either.

A third approach involves the conformal boundary of Minkowski space, in particular past and future null infinity. It has been known for a long time that the asymptotic symmetry group of this boundary is the so-called BMS group, an infinite dimensional group. In the spirit of AdS/CFT, one might try to look for a theory that carries representations of this large group. This is also quite problematic, see [5] for a recent discussion.

A final approach involves slicing of Minkowski space in Anti-de Sitter and de Sitter slices. These slices are given by the equation  $\eta_{\mu\nu}x^\mu x^\nu = r$ , where  $r < 0$  gives rise to AdS slices and  $r > 0$  gives rise to dS slices. The case  $r = 0$  corresponds to the light-cone. The idea is now to apply holography to each slice separately and then to combine the results. In this way one obtains a holographic dual theory that lives on the boundary of the light-cone, i.e. in two dimensions less, and which has infinitely many degrees of freedom.<sup>9</sup> Although various miracles happen [20] it remains to be seen whether these have essentially a kinematic origin, or whether they reveal some true holographic nature of Minkowski space.

## References

- [1] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, Large  $N$  field theories, string theory and gravity, *Phys. Rep.* 323 (2000), 183–386 [arXiv:hep-th/9905111].
- [2] S. Aminneborg, I. Bengtsson, D. Brill, S. Holst and P. Peldan, Black holes and wormholes in  $2 + 1$  dimensions, *Classical Quantum Gravity* 15 (1998), 627–644 [arXiv:gr-qc/9707036]; S. Aminneborg, I. Bengtsson, S. Holst and P. Peldan, Making anti-de Sitter black holes, *Classical Quantum Gravity* 13 (1996), 2707–2714 [arXiv:gr-qc/9604005]; M. Banados, Constant curvature black holes, *Phys. Rev. D* 57 (1998), 1068–1072 [arXiv:gr-qc/9703040]; K. Krasnov, Analytic continuation for asymptotically AdS 3D gravity, *Classical Quantum Gravity* 19 (2002), 2399–2424 [arXiv:gr-qc/0111049]; D. R. Brill, Multi-black-hole geometries in  $(2 + 1)$ -dimensional gravity, *Phys. Rev. D* 53 (1996), 4133–4137 [arXiv:gr-qc/9511022].
- [3] M. T. Anderson, Geometric aspects of the AdS/CFT correspondence, in: *AdS-CFT Correspondence: Einstein metrics and their conformal boundaries*, ed. by Olivier Biquard, IRMA Lect. Math. Theor. Phys. 8, European Math. Soc. Publishing House, Zürich 2005, 1–31 [arXiv:hep-th/0403087].

---

<sup>9</sup>One problem is that the dS/AdS slicing of Minkowski space does not really survive small deformations. In  $3+1$  dimensions, there are no deformations which give self-similar slices (i.e. slices which only differ from each other by rescaling). Such deformations may exist in higher dimensions, however the global spaces always have a singularity. We thank M. Anderson for his comment regarding this. See [3] for more details.

- [4] M. T. Anderson, P. T. Chrusciel and E. Delay, Non-trivial, static, geodesically complete vacuum space-times with a negative cosmological constant, *J. High Energy Phys.* 10 (2002), 063, 1–27.
- [5] G. Arcioni and C. Dappiaggi, Exploring the holographic principle in asymptotically flat spacetimes via the BMS group, *Nucl. Phys. B* 674 (2003), 553–592 [arXiv:hep-th/0306142]; Holography in asymptotically flat space-times and the BMS group, arXiv:hep-th/0312186.
- [6] V. Balasubramanian, J. de Boer and D. Minic, Mass, entropy and holography in asymptotically de Sitter spaces, *Phys. Rev. D* 65 (2002), 123508 [arXiv:hep-th/0110108].
- [7] V. Balasubramanian and T. S. Levi, Beyond the veil: Inner horizon instability and holography, arXiv:hep-th/0405048.
- [8] V. Balasubramanian, A. Naqvi and J. Simon, A multi-boundary AdS orbifold and DLCQ holography: A universal holographic arXiv:hep-th/0311237.
- [9] V. Balasubramanian and S. F. Ross, The dual of nothing, *Phys. Rev. D* 66 (2002), 086002 [arXiv:hep-th/0205290].
- [10] M. Banados, M. Henneaux, C. Teitelboim and J. Zanelli, Geometry of the  $(2 + 1)$  black hole, *Phys. Rev. D* 48 (1993), 1506–1525 [arXiv:gr-qc/9302012].
- [11] J. L. F. Barbon and E. Rabinovici, Long time scales and eternal black holes, *Fortschr. Phys.* 52 (2004), 642–649 [arXiv:hep-th/0403268].
- [12] J. D. Bekenstein, Black Holes And Entropy, *Phys. Rev. D* 7 (1973), 2333–2346; Generalized second law of thermodynamics in black hole physics, *Phys. Rev. D* 9 (1974), 3292.
- [13] L. Bers, Simultaneous Uniformization, *Bull. Amer. Math. Soc.* 66 (1960), 94–97; L. Bers, Spaces of Kleinian groups, in: *Several complex variables*, I (University of Maryland, College Park, Md., 1970), Lecture Notes in Math. 155, Springer-Verlag 1970, 9–34.
- [14] D. Birmingham and M. Rinaldi, Bubbles in anti-de Sitter space, *Phys. Lett. B* 544 (2002), 316–320 [arXiv:hep-th/0205246].
- [15] D. Birmingham, I. Sachs and S. N. Solodukhin, Relaxation in conformal field theory, Hawking–Page transition, and quasinormal/normal modes, *Phys. Rev. D* 67 (2003), 104026 [arXiv:hep-th/0212308].
- [16] D. R. Brill, J. Louko and P. Peldan, Thermodynamics of  $(3 + 1)$ -dimensional black holes with toroidal or higher genus horizons, *Phys. Rev. D* 56 (1997), 3600–3610 [arXiv:gr-qc/9705012].
- [17] M. I. Cai and G. J. Galloway, Boundaries of zero scalar curvature in the AdS/CFT correspondence, *Adv. Theor. Math. Phys.* 3 (1999), 1769–1783 [arXiv:hep-th/0003046].
- [18] R. Clarkson, A. M. Ghezelbash and R. B. Mann, Entropic  $N$ -bound and maximal mass conjectures violation in four-dimensional Taub–Bolt(NUT)-dS spacetimes, *Nucl. Phys. B* 674 (2003), 329–364 [arXiv:hep-th/0307059]; M. Anderson, On the structure of asymptotically de Sitter and anti-de Sitter spaces, arXiv:hep-th/0407087.
- [19] J. de Boer, Large  $N$  Elliptic Genus and AdS/CFT Correspondence, *J. High Energy Phys.* 05 (1999), 017 [arXiv:hep-th/9812240].

- [20] J. de Boer and S. N. Solodukhin, A holographic reduction of Minkowski space-time, *Nucl. Phys. B* 665 (2003), 545 [arXiv:hep-th/0303006]; S. N. Solodukhin, Reconstructing Minkowski space-time, arXiv:hep-th/0405252.
- [21] R. Bousso, A Covariant Entropy Conjecture, *J. High Energy Phys.* 07 (1999), 004 [arXiv:hep-th/9905177].
- [22] R. Dijkgraaf, J. M. Maldacena, G. W. Moore and E. Verlinde, A black hole farey tail, [arXiv:hep-th/0005003].
- [23] L. Fidkowski, V. Hubeny, M. Kleban and S. Shenker, The black hole singularity in AdS/CFT, *J. High Energy Phys.* 02 (2004), 014 [arXiv:hep-th/0306170].
- [24] H. Friedrich, On the existence of  $n$ -geodesically complete or future complete solutions of Einstein's field equations with smooth asymptotic structure, *Comm. Math. Phys.* 107 (1986), 587–609.
- [25] G. J. Galloway, K. Schleich, D. Witt and E. Woolgar, The AdS/CFT correspondence conjecture and topological censorship, *Phys. Lett. B* 505 (2001), 255–262 [arXiv:hep-th/9912119].
- [26] G. J. Galloway, S. Surya and E. Woolgar, A uniqueness theorem for the anti-de Sitter soliton, *Phys. Rev. Lett.* 88 (2002), 101102 [arXiv:hep-th/0108170]; G. J. Galloway, S. Surya and E. Woolgar, On the geometry and mass of static, asymptotically AdS spacetimes, and the uniqueness of the AdS soliton, *Comm. Math. Phys.* 241 (2003), 1–25 [arXiv:hep-th/0204081].
- [27] A. M. Ghezelbash and R. B. Mann, Nutty bubbles, *J. High Energy Phys.* 09(2002), 045 [arXiv:hep-th/0207123].
- [28] P. H. Ginsparg and M. J. Perry, Semiclassical Perdurance of de Sitter Space, *Nucl. Phys. B* 222 (1983), 245–268 ; J. C. Niemeyer and R. Bousso, The nonlinear evolution of de Sitter space instabilities, *Phys. Rev. D* 62 (2000), 023503 [arXiv:gr-qc/0004004]; R. Bousso, O. DeWolfe and R. C. Myers, Unbounded entropy in spacetimes with positive cosmological constant, *Found. Phys.* 33 (2003), 297–321 [arXiv:hep-th/0205080].
- [29] S. Giusto, S. D. Mathur and A. Saxena, Dual geometries for a set of 3-charge microstates, arXiv:hep-th/0405017; S. Giusto, S. D. Mathur and A. Saxena, 3-charge geometries and their CFT duals, arXiv:hep-th/0406103.
- [30] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Gauge theory correlators from non-critical string theory, *Phys. Lett. B* 428 (1998), 105–114 [arXiv:hep-th/9802109].
- [31] S. Gukov, Three-dimensional quantum gravity, Chern–Simons theory, and the A-polynomial, arXiv:hep-th/0306165.
- [32] S. Gukov, E. Martinec, G. Moore and A. Strominger, Chern–Simons gauge theory and the AdS(3)/CFT(2) correspondence, arXiv:hep-th/0403225.
- [33] S. W. Hawking, Breakdown of Predictability in Gravitational Collapse, *Phys. Rev. D* 14 (1976), 2460–2473.
- [34] S. W. Hawking, C. J. Hunter and D. N. Page, Nut charge, anti-de Sitter space and entropy, *Phys. Rev. D* 59 (1999), 044033 [arXiv:hep-th/9809035].



- [35] S. W. Hawking and D. N. Page, Thermodynamics of black holes in anti-de Sitter space, *Comm. Math. Phys.* 87 (1983), 577–588.
- [36] M. Henneaux, L. Maoz and A. Schwimmer, Asymptotic dynamics and asymptotic symmetries of three-dimensional extended AdS supergravity, *Ann. Physics* 282 (2000), 31–66 [arXiv:hep-th/9910013].
- [37] G. T. Horowitz and J. Maldacena, The black hole final state, *J. High Energy Phys.* 02 (2004), 008 [arXiv:hep-th/0310281].
- [38] G. T. Horowitz and R. C. Myers, The AdS/CFT correspondence and a new positive energy conjecture for general relativity, *Phys. Rev. D* 59 (1999), 026005 [arXiv:hep-th/9808079].
- [39] M. Kleban, M. Porrati and R. Rabadan, Poincare recurrences and topological diversity, arXiv:hep-th/0407192.
- [40] K. Krasnov, Holography and Riemann surfaces, *Adv. Theor. Math. Phys.* 4 (2000), 929–979 [arXiv:hep-th/0005106].
- [41] P. Kraus, H. Ooguri and S. Shenker, Inside the horizon with AdS/CFT, *Phys. Rev. D* 67 (2003), 124022 [arXiv:hep-th/0212277].
- [42] D. Kutasov and N. Seiberg, Number of degrees of freedom, density of states and tachyons in strong theory and CFT, *Nucl. Phys. B* 358 (1991), 600–618; E. J. Martinec, Conformal field theory, geometry, and entropy, arXiv:hep-th/9808021; S. Carlip, What we don't know about BTZ black hole entropy, *Classical Quantum Gravity* 15 (1998), 3609–3625 [arXiv:hep-th/9806026].
- [43] T. S. Levi and S. F. Ross, Holography beyond the horizon and cosmic censorship, *Phys. Rev. D* 68 (2003), 044005 [arXiv:hep-th/0304150].
- [44] O. Lunin, Adding momentum to D1–D5 system, [arXiv:hep-th/0404006].
- [45] O. Lunin, J. M. Maldacena and L. Maoz, Gravity solutions for the D1–D5 system with angular momentum, [arXiv:hep-th/0212210].
- [46] O. Lunin and S. D. Mathur, Metric of the multiply wound rotating string, *Nucl. Phys. B* 610 (2001), 49–76 [arXiv:hep-th/0105136]; AdS/CFT duality and the black hole information paradox, *Nucl. Phys. B* 623 (2002), 342–394 [arXiv:hep-th/0109154]; Rotating deformations of  $\text{AdS}(3) \times S^1$ , the orbifold CFT and strings in the pp-wave limit, *Nucl. Phys. B* 642 (2002), 91–113 [arXiv:hep-th/0206107]; O. Lunin, S. D. Mathur and A. Saxena, What is the gravity dual of a chiral primary?, arXiv:hep-th/0211292.
- [47] J. M. Maldacena, The large  $N$  limit of superconformal field theories and supergravity, *Adv. Theor. Math. Phys.* 2 (1998), 231–252; [*Internat. J. Theoret. Phys.* 38 (1999), 1113–1133] [arXiv:hep-th/9711200].
- [48] J. M. Maldacena, Eternal black holes in anti-de-Sitter, *J. High Energy Phys.* 04 (2003), 021 [arXiv:hep-th/0106112].
- [49] J. Maldacena and L. Maoz, Wormholes in AdS, *J. High Energy Phys.* 02 (2004), 053 [arXiv:hep-th/0401024].
- [50] D. Mateos and P. K. Townsend, Supertubes, *Phys. Rev. Lett.* 87 (2001), 011602 [arXiv:hep-th/0103030]; R. Emparan, D. Mateos and P. K. Townsend, Supergravity supertubes, *J. High Energy Phys.* 07 (2001), 011 [arXiv:hep-th/0106012]; D. Mateos, S. Ng and

- P. K. Townsend, Tachyons, supertubes and brane/anti-brane systems, *J. High Energy Phys.* 03 (2002), 016 [arXiv:hep-th/0112054]; D. Mateos, S. Ng and P. K. Townsend, Supercurves, *Phys. Lett. B* 538 (2002), 366–374 [arXiv:hep-th/0204062].
- [51] B. McInnes, Quintessential Maldacena–Maoz cosmologies, *J. High Energy Phys.* 04 (2004), 036 [arXiv:hep-th/0403104]; B. McInnes, Answering a basic objection to Bang/Crunch holography, arXiv:hep-th/0407189.
- [52] D. N. Page, Phase transitions for gauge theories on tori from the AdS/CFT correspondence, arXiv:hep-th/0205001.
- [53] J. Polchinski, S-matrices from AdS spacetime, arXiv:hep-th/9901076.
- [54] S. N. Solodukhin, Can black hole relax unitarily?, arXiv:hep-th/0406130.
- [55] S. Surya, K. Schleich and D. M. Witt, Phase transitions for flat anti-de Sitter black holes, *Phys. Rev. Lett.* 86 (2001), 5231–5234 [arXiv:hep-th/0101134].
- [56] W. Thurston, Three-dimensional manifolds, Kleinian groups and hyperbolic space forms, *Bull. Amer. Math. Soc. (N.S.)* 6 (1982), 357–381; W. D. Neumann and D. Zagier, Volumes of hyperbolic three-manifolds, *Topology* 24 (1985), 307–332; T. Yoshida, The  $\eta$ -invariant of hyperbolic 3-manifolds, *Invent. Math.* 81 (1985), 473–514.
- [57] L. Vanzo, Black holes with unusual topology, *Phys. Rev. D* 56 (1997), 6475–6483 [arXiv:gr-qc/9705004].
- [58] E. Witten,  $(2 + 1)$ -dimensional gravity as an exactly soluble system, *Nucl. Phys. B* 311 (1988), 46–78.
- [59] E. Witten, Anti-de Sitter space and holography, *Adv. Theor. Math. Phys.* 2 (1998), 253–291 [arXiv:hep-th/9802150].
- [60] E. Witten, Anti-de Sitter space, thermal phase transition, and confinement in gauge theories, *Adv. Theor. Math. Phys.* 2 (1998), 505–532 [arXiv:hep-th/9803131].
- [61] E. Witten, Quantum gravity in de Sitter space, arXiv:hep-th/0106109; A. Strominger, The dS/CFT correspondence, *J. High Energy Phys.* 10 (2001), 034 [arXiv:hep-th/0106113]; V. Balasubramanian, J. de Boer and D. Minic, Notes on de Sitter space and holography, *Classical Quantum Gravity* 19 (2002), 5655–5700; Exploring de Sitter space and holography, *Ann. Physics* 303 (2003), 59–116 [arXiv:hep-th/0207245]; M. Spradlin and A. Volovich, Vacuum states and the S-matrix in dS/CFT, *Phys. Rev. D* 65 (2002), 104037 [arXiv:hep-th/0112223]; N. Goheer, M. Kleban and L. Susskind, The trouble with de Sitter space, *J. High Energy Phys.* 07 (2003), 056 [arXiv:hep-th/0212209]; T. Banks, A critique of pure string theory: Heterodox opinions of diverse dimensions, arXiv:hep-th/0306074.
- [62] E. Witten and S. T. Yau, Connectedness of the boundary in the AdS/CFT correspondence, *Adv. Theor. Math. Phys.* 3 (1999), 1635–1655 [arXiv:hep-th/9910245].



# The ambient obstruction tensor and $Q$ -curvature

*C. Robin Graham and Kengo Hirachi\**

*Department of Mathematics, University of Washington, Box 354350  
Seattle, WA 98195-4350, U.S.A.  
email: robin@math.washington.edu*

*Graduate School of Mathematical Sciences, University of Tokyo  
3-8-1 Komaba, Meguro, Tokyo 153-8914, Japan  
email: hirachi@ms.u-tokyo.ac.jp*

## 1 Introduction

The Bach tensor is a basic object in four-dimensional conformal geometry. It is a conformally invariant trace-free symmetric 2-tensor involving 4 derivatives of the metric which is of particular interest because it vanishes for metrics which are conformal to Einstein metrics, and because it arises as the first variational derivative of the conformally invariant Lagrangian  $\int |W|^2$ , where  $W$  denotes the Weyl tensor. A generalization of the Bach tensor to higher even dimensional manifolds was indicated in [FG1]. This “ambient obstruction tensor”, which, suitably normalized, we denote by  $\mathcal{O}_{ij}$ , is also a trace-free symmetric 2-tensor which is conformally invariant and vanishes for conformally Einstein metrics. It involves  $n$  derivatives of the metric on a manifold of even dimension  $n \geq 4$ . In this paper we give the details of the derivation and basic properties of the obstruction tensor and provide a characterization generalizing the variational characterization of the Bach tensor in four dimensions. We also give an invariant-theoretic classification of conformally invariant tensors up to invariants which are quadratic and higher in curvature which illuminates the fundamental nature of the obstruction tensor.

Our higher dimensional substitute for  $\int |W|^2$  is the integral of Branson’s  $Q$ -curvature ([B]). This  $Q$ -curvature is a scalar quantity defined on even-dimensional Riemannian (or pseudo-Riemannian) manifolds. It is not a pointwise conformal invariant like  $|W|^2$ , but it does have a simple transformation law under conformal

---

\*The work of the first author was partially supported by NSF grant DMS-0204480. The work of the second author was supported by Grant-in-Aid for Scientific Research, JSPS. The authors are grateful to the Erwin Schrödinger Institute for hospitality during the writing of this paper.

change which implies that its integral over a compact manifold is a conformal invariant. In dimension 4, one has  $6Q = -\Delta R + R^2 - 3|\text{Ric}|^2$ , where  $R$  denotes the scalar curvature and  $\Delta = \nabla^i \nabla_i$ . Since the Pfaffian in 4 dimensions is a multiple of  $R^2 - 3|\text{Ric}|^2 + \frac{3}{2}|W|^2$ , it follows that  $\int Q$  is a linear combination of the Euler characteristic and  $\int |W|^2$ , so the variational derivatives of  $\int Q$  and  $\int |W|^2$  are multiples of one another. It follows from a result announced by Alexakis [Al] that also in higher dimensions,  $\int Q$  is a linear combination of the Euler characteristic and the integral of a pointwise conformal invariant. However explicit formulae are not available.

Our variational characterization is:

**Theorem 1.1.** *If  $g^t$  is a 1-parameter family of metrics on a compact manifold  $M$  of even dimension  $n \geq 4$ , then*

$$\left( \int_M Q \, dv \right)' = (-1)^{n/2} \frac{n-2}{2} \int_M \mathcal{O}_{ij} \dot{g}^{ij} \, dv,$$

where  $' = \partial_t|_{t=0}$  and  $\mathcal{O}_{ij}$  and  $dv$  on the right-hand side are with respect to  $g^0$ .

In [FG1],  $\mathcal{O}_{ij}$  arose as the obstruction to the existence of a smooth formal power series solution for the ambient metric associated to the given conformal structure, a Ricci-flat metric in 2 higher dimensions homogeneous with respect to dilations. As described in [FG1], the ambient metric is equivalent to a Poincaré metric, a metric in 1 higher dimension with constant negative Ricci curvature having the given conformal structure as conformal infinity, and the obstruction tensor may alternately be viewed as obstructing smooth formal power series solutions for a Poincaré metric. It is the latter formulation that we use in this paper, for it is the Poincaré metric that provides the link between  $Q$ -curvature and the obstruction tensor. Specifically, we use the result of [GZ] that the integral of the  $Q$ -curvature is equal to a multiple of the log term in the volume expansion of a Poincaré metric. We then calculate the variation of the log term coefficient by a simplified version of the method of Anderson [An] for expressing the variation of volume as a boundary integral. A different calculation of the first variation of the log coefficient in the volume expansion is given in [HSS].

The existence of the obstruction tensor gives rise to the questions of whether there are other conformally invariant tensors lurking in the shadows, and whether there is some kind of odd-dimensional analogue. Of course, one may construct further invariants from known ones by taking tensor products and contracting. However, the following result shows that up to quadratic and higher terms in curvature, the Weyl tensor (or Cotton tensor in dimension 3) and the obstruction tensor are the only irreducible conformally invariant tensors.

**Theorem 1.2.** *A conformally invariant irreducible natural tensor of  $n$ -dimensional oriented Riemannian manifolds is equivalent modulo a conformally invariant natural tensor of degree at least 2 in curvature with a multiple of one of the following:*

- $n = 3$ : the Cotton tensor  $C_{ijk} = P_{ij,k} - P_{ik,j}$ ,

- $n = 4$ : the self-dual or anti-self dual Weyl tensor  $W_{ijkl}^\pm$  or the Bach tensor  $B_{ij} = \mathcal{O}_{ij}$ ,
- $n \geq 5$  odd: the Weyl tensor  $W_{ijkl}$ ,
- $n \geq 6$  even: the Weyl tensor  $W_{ijkl}$  or the obstruction tensor  $\mathcal{O}_{ij}$ .

Here the trace-modified Ricci tensor  $P_{ij}$  is defined by

$$(n-2)P_{ij} = R_{ij} - \frac{R}{2(n-1)}g_{ij}, \quad (1.1)$$

and the terminology used in the statement of the Theorem is explained in §4. Theorem 1.2 is an easy consequence of the classification of conformally invariant linear differential operators on the sphere due to Boe–Collingwood ([BC]).

In §2 we show how to derive the obstruction tensor in terms of a Poincaré metric and establish its basic properties. In §3 we prove Theorem 1.1 and in §4 we prove Theorem 1.2.

We are grateful to Mike Eastwood and Andi Čap for helpful discussions concerning the Boe–Collingwood classification of invariant operators.

## 2 The obstruction tensor

In this section we provide the details of the background about the obstruction tensor. We show how it arises as the obstruction to the existence of a smooth formal power series solution for a Poincaré metric associated to the given conformal structure and derive its properties from this characterization. We also show that this definition may be reformulated in terms of a formal solution to one higher order involving a log term.

Let  $M$  be a manifold of dimension  $n \geq 3$  with smooth conformal structure  $[g]$  of signature  $(p, q)$  and let  $X$  be  $n+1$ -manifold with boundary  $M$ . All our considerations in this section are local near a point of  $M$ . We are interested in conformally compact metrics  $g_+$  of signature  $(p+1, q)$  on  $X$  with conformal infinity  $[g]$ . This means that if  $x$  is a smooth defining function for  $M$ , then  $x^2g_+$  is a smooth (to some order) metric on  $X$  with  $x^2g_+|_{TM} \in [g]$ . If  $n$  is odd, then for any conformal class  $[g]$  there are metrics  $g_+$  with  $x^2g_+$  a formal smooth power series such that  $\text{Ric } g_+ = -ng_+$  to infinite order. However if  $n$  is even, the obstruction tensor obstructs the existence of formal smooth solutions at order  $n-2$ . (Throughout this paper, when we say that a tensor is  $O(x^s)$ , we mean that all components of the tensor are  $O(x^s)$  in a smooth coordinate system on  $X$ .)<sup>1</sup>

**Theorem 2.1.** *If  $n \geq 4$  is even, there exists a metric  $g_+$  with  $x^2g_+$  smooth such that  $g_+$  has  $[g]$  as conformal infinity and  $\text{Ric } g_+ + ng_+ = O(x^{n-2})$ .  $g_+$  is unique modulo*

<sup>1</sup>One could alternately consider metrics of signature  $(p, q+1)$  for which  $\text{Ric } g_+ = ng_+$ . This formulation is equivalent to ours via the change  $g_+ \rightarrow -g_+$ .

$O(x^{n-2})$  up to a diffeomorphism of  $X$  which restricts to the identity on  $M$ . The tensor  $\text{tf}(x^{2-n}(\text{Ric } g_+ + ng_+)|_{TM})$  on  $M$  is independent of the choice of such  $g_+$ , where  $\text{tf}$  denotes the trace-free part with respect to  $[g]$ . We define the obstruction tensor

$$\mathcal{O} = c_n \text{tf}(x^{2-n}(\text{Ric } g_+ + ng_+)|_{TM}), \quad c_n = \frac{2^{n-2}(n/2 - 1)!^2}{n - 2}. \quad (2.1)$$

Then  $\mathcal{O}_{ij}$  has the properties:

1.  $\mathcal{O}$  is a natural tensor invariant of the metric  $g = x^2 g_+|_{TM}$ ; i.e. in local coordinates the components of  $\mathcal{O}$  are given by universal polynomials in the components of  $g$ ,  $g^{-1}$  and the curvature tensor of  $g$  and its covariant derivatives. The expression for  $\mathcal{O}_{ij}$  takes the form

$$\begin{aligned} \mathcal{O}_{ij} &= \Delta^{n/2-2} (P_{ij,k}{}^k - P_k{}^k{}_{,ij}) + \text{lots} \\ &= (3-n)^{-1} \Delta^{n/2-2} W_{kijl}{}^{kl} + \text{lots}, \end{aligned} \quad (2.2)$$

where  $\Delta = \nabla^i \nabla_i$ ,  $W$  denotes the Weyl tensor of  $g$ , and  $\text{lots}$  denotes quadratic and higher terms in curvature involving fewer derivatives.

2. One has

$$\mathcal{O}_i{}^i = 0 \quad \text{and} \quad \mathcal{O}_{ij}{}^{,j} = 0.$$

3.  $\mathcal{O}_{ij}$  is conformally invariant of weight  $2-n$ ; i.e. if  $0 < \Omega \in C^\infty(M)$  and  $\hat{g}_{ij} = \Omega^2 g_{ij}$ , then  $\hat{\mathcal{O}}_{ij} = \Omega^{2-n} \mathcal{O}_{ij}$ .
4. If  $g_{ij}$  is conformal to an Einstein metric, then  $\mathcal{O}_{ij} = 0$ .

*Proof.* There are discussions of the asymptotics of Poincaré metrics in the literature, but we provide a self-contained treatment.

We shall work with metrics in a normal form. Lemma 5.2 and the subsequent paragraph in [GL] imply that if one is given a conformally compact metric  $g_+$  which is asymptotically Einstein in the sense that  $\text{Ric } g_+ + ng_+ = O(x^{-1})$  and a representative metric  $g \in [g]$ , there is an identification of a neighborhood of  $M$  in  $X$  with  $M \times [0, \epsilon)$  such that  $g_+$  takes the form

$$g_+ = x^{-2}(dx^2 + g_x) \quad (2.3)$$

for a 1-parameter family  $g_x$  of metrics on  $M$  with  $g_0 = g$ .

It is straightforward to calculate  $E = \text{Ric } g_+ + ng_+$  for  $g_+$  of the form (2.3). We use Greek indices to label objects on  $X$ , Latin indices for  $M$ , and 0 for  $\partial_x$  so that in an identification  $X \cong M \times [0, \epsilon)$  as above, a Greek index  $\alpha$  corresponds to a pair  $(i, 0)$ . One obtains:

$$\begin{aligned} 2x E_{ij} &= -x g''_{ij} + x g^{kl} g'_{ik} g'_{jl} - \frac{x}{2} g^{kl} g'_{kl} g'_{ij} \\ &\quad + (n-1) g'_{ij} + g^{kl} g'_{kl} g_{ij} + 2x \text{Ric}(g_x)_{ij} \end{aligned} \quad (2.4)$$

$$E_{i0} = \frac{1}{2}g^{kl}(\nabla_l g'_{ik} - \nabla_i g'_{kl}) \quad (2.5)$$

$$E_{00} = -\frac{1}{2}g^{kl}g''_{kl} + \frac{1}{4}g^{kl}g^{pq}g'_{kp}g'_{lq} + \frac{1}{2}x^{-1}g^{kl}g'_{kl}, \quad (2.6)$$

where  $'$  denotes  $\partial_x$ , we have suppressed the subscript on  $g_x$ , and  $\nabla$  and  $\text{Ric}$  denote the Levi-Civita connection and Ricci curvature of  $g_x$  for fixed  $x$ .

One can determine the derivatives of  $g_x$  inductively to solve the equation  $E_{ij} = O(x^{n-2})$ , beginning with the prescription  $g_0 = g$ . Differentiating (2.4)  $s-1$  times and setting  $x=0$  gives

$$\begin{aligned} \partial_x^{s-1}(2xE_{ij})|_{x=0} &= (n-s)\partial_x^s g_{ij} + g^{kl}\partial_x^s g_{kl}g_{ij} \\ &\quad + (\text{terms involving } \partial_x^k g_{ij} \text{ with } k < s) \end{aligned} \quad (2.7)$$

For  $s \neq n, 2n$ , the operator  $\eta_{ij} \rightarrow (n-s)\eta_{ij} + g^{kl}\eta_{kl}g_{ij}$  is invertible on symmetric 2-tensors at each point of  $M$ . It follows inductively that one uniquely obtains a metric  $g_+ \bmod O(x^{n-2})$  of the form (2.3) by the requirement  $E_{ij} = O(x^{n-2})$ . Moreover, the derivatives of  $g_x$  at  $x=0$  of order less than  $n$  are all natural tensor invariants of the initial representative metric  $g$ .

The vanishing of the remaining components of  $E$  to the correct order is deduced via the Bianchi identity. The Bianchi identity for Ricci curvature of  $g_+$  states

$$g_+^{\alpha\beta}\nabla_\gamma^+ E_{\alpha\beta} = 2g_+^{\alpha\beta}\nabla_\alpha^+ E_{\beta\gamma},$$

where  $\nabla^+$  denotes the Levi-Civita connection of  $g_+$ . Taking separately  $\gamma=0$  and  $\gamma=i$  and writing this in terms of the connection  $\nabla$  of  $g_x$  gives the following two equations:

$$g^{jk}E'_{jk} = 2\nabla^j E_{j0} + (\partial_x + g^{jk}g'_{jk} - 2(n-1)x^{-1})E_{00} \quad (2.8)$$

$$\partial_i E_{00} + \nabla_i E_j{}^j - 2\nabla^j E_{ij} = 2(\partial_x + \frac{1}{2}g^{jk}g'_{jk} - (n-1)x^{-1})E_{i0}. \quad (2.9)$$

We claim that  $E_{00} = O(x^{n-2})$  and  $E_{i0} = O(x^{n-1})$ . These follow by induction on the statement that  $E_{00} = O(x^{s-1})$  and  $E_{i0} = O(x^s)$  for  $0 \leq s \leq n-1$ . The case  $s=0$  is immediate from (2.6) and (2.5). Suppose the statement is true for some  $s, s \leq n-2$ . Write  $E_{00} = \lambda x^{s-1}$  and recall  $E_{ij} = O(x^{n-2})$ . In (2.8), we have  $g^{jk}E'_{jk} = O(x^{n-3}) = O(x^{s-1})$ ,  $\nabla^j E_{j0} = O(x^s)$ , and  $g^{jk}g'_{jk} = O(1)$ . Calculating (2.8) mod  $O(x^{s-1})$  thus gives  $(s-2n+1)\lambda x^{s-2} = O(x^{s-1})$ , which implies  $\lambda = O(x)$  so  $E_{00} = O(x^s)$  as desired. Now write  $E_{i0} = \mu_i x^s$  and calculate (2.9) mod  $O(x^s)$ . One obtains similarly  $(s+1-n)\mu_i x^{s-1} = O(x^s)$ . Since  $s \leq n-2$  it follows that  $\mu_i = O(x)$  so  $E_{i0} = O(x^{s+1})$ , completing the induction.

This proves the first sentence of the statement of Theorem 2.1: existence of a formal solution to order  $n-2$ . The second sentence, uniqueness of  $g_+$  up to diffeomorphism, follows from the fact that any metric  $g_+$  can be put into the form (2.3) by a



diffeomorphism together with the uniqueness of the determination of  $g_+$  in the form (2.3) as above. The definition (2.1) of  $\mathcal{O}$  depends on a choice of defining function  $x$  to first order, equivalently on the choice of a conformal representative, but is otherwise diffeomorphism invariant. So in order to establish the independence of  $\mathcal{O}$  on the freedom in  $g_+$  at order  $n - 2$  and the naturality of  $\mathcal{O}$ , it suffices to consider  $g_+$  of the form (2.3). These conclusions now follow from (2.7): taking  $s = n$  shows that the trace-free part of  $x^{2-n} E_{ij}|_{x=0}$  is given purely in terms of the previously determined terms.

The tensor  $\mathcal{O}_{ij}$  is trace-free by definition. The fact that  $\mathcal{O}_{ij,j} = 0$  can be established by consideration of (2.8) and (2.9) as follows. Consider the approximately Einstein metric  $g_+ \bmod O(x^{n-2})$  constructed inductively above which satisfies  $E_{ij} = O(x^{n-2})$ ,  $E_{i0} = O(x^{n-1})$ ,  $E_{00} = O(x^{n-2})$ . Although vanishing of the trace-free part of  $E_{ij}$  at order  $n - 2$  is obstructed by  $\mathcal{O}_{ij}$ , one sees from (2.7) that one can solve for the trace to ensure that  $g^{ij} E_{ij} = O(x^{n-1})$ . This is sufficient to allow one to conclude exactly as above from (2.8) that  $E_{00} = O(x^{n-1})$ . Now consider (2.9). One finds this time that the right-hand side is already  $O(x^{n-1})$ . Substituting  $E_{ij} = \mathcal{O}_{ij} x^{n-2} \bmod O(x^{n-1})$  and calculating  $\bmod O(x^{n-1})$  gives  $\mathcal{O}_{ij,j} = 0$  as desired.

The conformal invariance of  $\mathcal{O}_{ij}$  follows immediately from its definition: the rescaling  $\hat{g}_{ij} = \Omega^2 g_{ij}$  corresponds to  $\hat{x} = \Omega x + O(x^2)$ , which by (2.1) gives  $\hat{\mathcal{O}}_{ij} = \Omega^{2-n} \mathcal{O}_{ij}$ . The fact that the same tensor arises when calculated in the normal forms determined by different conformal representatives is implicit in the invariance of the definition under diffeomorphisms.

The vanishing of  $\mathcal{O}_{ij}$  for conformally Einstein metrics follows from the fact that for  $g$  Einstein, one can write down an explicit solution for  $g_+$ . It is well known that if  $\text{Ric}(g) = 4\lambda(n - 1)g$ , then the metric  $g_+ = x^{-2}(dx^2 + (1 - \lambda x^2)^2 g)$  satisfies  $\text{Ric}(g_+) = -ng_+$ . This is also easily checked directly using (2.4)–(2.6). In particular there is no obstruction to existence of a smooth formal solution at order  $n - 2$ , so it must be that  $\mathcal{O}_{ij} = 0$ .

To finish the proof of Theorem 2.1, it remains to derive the principal part of  $\mathcal{O}_{ij}$ . This can be done by keeping track of the leading term in the inductive derivation above. As described above, the derivatives  $\partial_x^s(g_x)|_{x=0}$  for  $1 \leq s \leq n - 1$  are determined inductively by setting  $E_{ij} = 0$  and differentiating in (2.4), and the obstruction  $\mathcal{O}_{ij}$  arises when trying to solve for  $\partial_x^n(g_x)|_{x=0}$ . Parity considerations show that these derivatives vanish for  $s$  odd. Differentiating (2.4) once gives  $g''_{ij}|_{x=0} = -2P_{ij}$ . Differentiating further and using the first variation of Ricci curvature

$$\dot{\text{Ric}}_{ij} = \frac{1}{2}(\dot{g}_{ik,j}{}^k + \dot{g}_{jk,i}{}^k - \dot{g}_{ij,k}{}^k - \dot{g}_k{}^k{}_{,ij})$$

and the Bianchi identity  $P_{ik,k} = P_k{}^k{}_{,i}$ , one determines inductively that

$$\partial_x^{2m} g_{ij}|_{x=0} = 2 \frac{3 \cdot 5 \cdot 7 \dots (2m - 1)}{(n - 4)(n - 6) \dots (n - 2m)} (\Delta^{m-2} P_k{}^k{}_{,ij} - \Delta^{m-1} P_{ij}) + \text{lots}$$

for  $2 \leq m < n/2$ . Using  $E_{ij} = c_n^{-1} x^{n-2} \mathcal{O}_{ij} \mod O(x^{n-1})$  in (2.4) and differentiating  $n-1$  times then gives the first line of (2.2). The second follows from the fact that  $W_{kijl,}{}^{kl} = (3-n)(P_{ij,k}{}^k - P_{ik,j}{}^k)$ .  $\square$

The proof of Theorem 2.1 gives an algorithm for the calculation of  $\mathcal{O}_{ij}$ . For  $n = 4, 6$ , carrying out the calculations gives the following explicit formulae. Define the Cotton and Bach tensors by:

$$C_{ijk} = P_{ij,k} - P_{ik,j} \quad \text{and} \quad B_{ij} = P_{ij,k}{}^k - P_{ik,j}{}^k - P^{kl} W_{kijl}.$$

Then when  $n = 4$  one has  $\mathcal{O}_{ij} = B_{ij}$  and when  $n = 6$  one has

$$\begin{aligned} \mathcal{O}_{ij} = & B_{ij,k}{}^k - 2W_{kijl} B^{kl} - 4P_k{}^k B_{ij} + 8P^{kl} C_{(ij)k,l} - 4C_i{}^k{}_l C_{ljk} \\ & + 2C_i{}^{kl} C_{jkl} + 4P_{k,l}{}^k C_{(ij)}{}^l - 4W_{kijl} P_m{}^k P^{ml}. \end{aligned}$$

If the obstruction tensor is nonzero, there are no formal smooth solutions to  $\text{Ric}(g_+) = -ng_+$  beyond order  $n-2$ . However, it is always possible to find solutions to all orders by including log terms in the expansion of  $g_+$ . For our purposes it will suffice to consider solutions to one higher order. The obstruction tensor  $\mathcal{O}_{ij}$  can then be characterized as the coefficient of the first log term.

**Theorem 2.2.** *In the setting of Theorem 2.1, there is a solution  $g_+$  to  $\text{Ric}(g_+) + ng_+ = O(x^{n-1} \log x)$  of the form  $g_+ = x^{-2}(dx^2 + g_x)$ , where  $g_x = h_x + r_x x^n \log x$  and  $h_x$  and  $r_x$  are smooth in  $x$ . The coefficient  $r_x$  is uniquely determined at  $x = 0$  and is given by  $nc_n r_0 = 2\mathcal{O}$ .*

*Proof.* Fix a metric  $g_+^0 = x^{-2}(dx^2 + g_x^0)$  with  $g_x^0$  smooth which solves  $E^0 = O(x^{n-2})$  as in Theorem 2.1. Set  $g_x = g_x^0 + r_x x^n \log x + s_x x^n$ . Substituting into (2.4) gives

$$\begin{aligned} 2xE_{ij} = & 2xE_{ij}^0 + g^{kl} r_{kl} g_{ij} (x^{n-1} \log x + x^{n-1}) \\ & - nr_{ij} x^{n-1} + ng^{kl} s_{kl} g_{ij} x^{n-1} \mod O(x^n \log x). \end{aligned}$$

It is required that this expression vanish mod  $O(x^n \log x)$ . The requirement that there be no  $x^{n-1} \log x$  term forces  $g^{kl} r_{kl}|_{x=0} = 0$ . Since  $c_n \text{tf}(x^{2-n} E_{ij}^0)|_{x=0} = \mathcal{O}_{ij}$ , we must have  $nc_n r_{ij}|_{x=0} = 2\mathcal{O}_{ij}$ . The trace of  $s_{ij}$  can be chosen to guarantee  $g^{ij}(x^{2-n} E_{ij}|_{x=0}) = 0$ ; the trace-free part of  $s_{ij}$  remains arbitrary. With these choices, if we set  $h_x = g_x^0 + s_x x^n \mod O(x^{n+1})$  and  $r_x = r \mod O(1)$ , we obtain  $g_+$  in the form required in the statement of the theorem satisfying  $E_{ij} = O(x^{n-1} \log x)$ .

In the proof of Theorem 2.1 it was shown that  $g_+^0$  satisfies  $E_{i0}^0 = O(x^{n-1})$ ,  $E_{00}^0 = O(x^{n-2})$ . It is evident from this and (2.5), (2.6) that  $g_+$  satisfies  $E_{i0} = O(x^{n-1} \log x)$ ,  $E_{00} = O(x^{n-2} \log x)$ . Arguing as in the proof of Theorem 2.1, one finds that (2.8) implies that in fact one has  $E_{00} = O(x^{n-1} \log x)$ , completing the proof.  $\square$

### 3 Proof of Theorem 1.1

For any metric  $g$  on  $M$ , we consider a metric  $g_+ = x^{-2}(dx^2 + g_x)$  on  $M \times (0, \epsilon)$  given by Theorem 2.2 which satisfies  $\text{Ric}(g_+) + ng_+ = O(x^{n-1} \log x)$ . Recall that  $g_x$  was determined only up to addition of a trace-free tensor in the coefficient of  $x^n$  and up to addition of terms of order greater than  $n$ . For definiteness, we specify  $g_x$  to be given by the finite expansion

$$g_x = g + g^{(2)}x^2 + (\text{even powers}) + \frac{2}{nc_n}\mathcal{O}x^n \log x + g^{(n)}x^n \quad (3.1)$$

where the  $g^{(2m)}$  for  $m < n/2$  are those coefficients derived in Theorem 2.1, and we take  $g^{(n)}$  to be the multiple of  $g$  determined in the proof of Theorem 2.2. Then the metric  $g_+$  is completely determined by  $g$ , and of course satisfies  $\text{Ric}(g_+) + ng_+ = O(x^{n-1} \log x)$ .

The proof of Theorem 1.1 depends on the volume expansion of an asymptotically Einstein metric; see [G]. The volume form of  $g_+$  is

$$dv_{g_+} = x^{-n-1} \left( \frac{\det g_x}{\det g} \right)^{1/2} dv_g dx.$$

From (3.1) and the fact that  $g^{ij}\mathcal{O}_{ij} = 0$ , it follows that

$$\left( \frac{\det g_x}{\det g} \right)^{1/2} = 1 + v^{(2)}x^2 + (\text{even powers}) + v^{(n)}x^n + \dots, \quad (3.2)$$

where the  $v^{(2j)}$  are locally determined invariant scalars given in terms of  $g$  and its curvature, and  $\dots$  denotes terms vanishing to higher order. Integrating, it follows that for fixed  $\epsilon_0$  we have the asymptotic expansion as  $\epsilon \rightarrow 0$

$$\begin{aligned} \text{Vol}_{g_+}(\{\epsilon < x < \epsilon_0\}) &= c_0\epsilon^{-n} + c_2\epsilon^{-n+2} + (\text{even powers}) \\ &\quad + c_{n-2}\epsilon^{-2} + L \log \frac{1}{\epsilon} + O(1), \end{aligned}$$

where  $c_{2j} = (n - 2j)^{-1} \int_M v^{(2j)} dv_g$  and  $L = \int_M v^{(n)} dv_g$ . The log term coefficient  $L$  is invariant under conformal rescalings of  $g$ ; see [G] for a proof.

The  $Q$ -curvature of a metric  $g$  was originally defined by Branson [B] in terms of the zero-th order term of the conformally invariant  $n$ -th power of the Laplacian  $P_n$  of [GJMS] by dimensional continuation. It is a scalar quantity with a particularly simple transformation law under conformal rescalings: if  $\hat{g} = e^{2\Upsilon}g$ , then  $e^{n\Upsilon}\hat{Q} = Q + P_n\Upsilon$ . Although  $Q$  is not pointwise conformally invariant, it follows from the facts that  $P_n$  is self-adjoint and annihilates constants that the integral  $\int Q dv$  over a compact manifold is a conformal invariant. Characterizations of the  $Q$ -curvature in terms of Poincaré metrics were given in [GZ] and [FG2], and in terms of the ambient metric in [FH]. We refer to [B] and these references for background about  $Q$ -curvature. The main fact

we will need here is the result of [GZ] (or see [FG2] for a simpler proof) that

$$\int Q dv = k_n L, \quad k_n = (-1)^{n/2} n(n-2) c_n. \quad (3.3)$$

According to (3.3), in order to calculate the variation of  $\int Q dv$  it suffices to compute  $\dot{L}$ . For this, we use a simplification of the method of Anderson [An] to rewrite the variation of volume as a boundary integral for variations through Einstein metrics with fixed scalar curvature. In our case we need to estimate the errors resulting from the fact that our metrics are only asymptotically Einstein.

**Lemma 3.1.** *Let  $g^t$  be a 1-parameter family of metrics on a compact manifold  $M$  and let  $g_+^t = x^{-2}(dx^2 + g_x^t)$  be the corresponding asymptotically Einstein metrics on  $M \times (0, \epsilon_0)$ , where for each  $t$ ,  $g_x^t$  is constructed from  $g^t$  as in (3.1). Set  $X_\epsilon = \{\epsilon < x < \epsilon_0\}$ . Then as  $\epsilon \rightarrow 0$  we have*

$$\text{Vol}_{g_+^t}(X_\epsilon) = \frac{\epsilon^{1-n}}{2n} \int_{x=\epsilon} \left( -\frac{1}{2} g^{ij} g^{kl} g'_{jl} \dot{g}_{ik} + x^{-1} g^{ij} \dot{g}_{ij} - (g^{ij} \dot{g}_{ij})' \right) dv_{g_\epsilon} + O(1). \quad (3.4)$$

On the right-hand side,  $'$  denotes  $\partial_x$  and  $\dot{\phantom{x}}$  denotes  $\partial_t|_{t=0}$  as usual, and all  $g = g_x^t$  are evaluated at  $x = \epsilon$ ,  $t = 0$  (after differentiation).

*Proof.* Since the metrics  $g_+^t$  are asymptotically Einstein, we have uniformly in  $t$  (and suppressing the  $t$ -dependence of  $g_+^t$ ):

$$\begin{aligned} \text{Ric}_{g_+} &= -n g_+ + O(x^{n-1} \log x), \\ R_{g_+} &= -n(n+1) + O(x^{n+1} \log x), \\ \dot{R}_{g_+} &= O(x^{n+1} \log x). \end{aligned}$$

Therefore  $\int_{X_\epsilon} \dot{R}_{g_+} dv_{g_+} = O(1)$  as  $\epsilon \rightarrow 0$ .

On the other hand, the usual formula for the first variation of scalar curvature gives

$$\begin{aligned} \dot{R}_{g_+} &= \dot{g}_{+\alpha\beta}^{\alpha\beta} - \dot{g}_{+\alpha}^{\alpha}{}_{,\beta}{}^{\beta} - \text{Ric}_{g_+}^{\alpha\beta} \dot{g}_{+\alpha\beta} \\ &= \dot{g}_{+\alpha\beta}^{\alpha\beta} - \dot{g}_{+\alpha}^{\alpha}{}_{,\beta}{}^{\beta} + n g_+^{\alpha\beta} \dot{g}_{+\alpha\beta} + O(x^{n+1} \log x), \end{aligned}$$

where the covariant derivatives are with respect to the Levi-Civita connection of  $g_+$  and indices are raised and lowered using  $g_+$ . Integrating gives

$$\int_{X_\epsilon} (\dot{g}_{+\alpha\beta}^{\alpha\beta} - \dot{g}_{+\alpha}^{\alpha}{}_{,\beta}{}^{\beta}) dv_{g_+} + 2n \int_{X_\epsilon} \dot{g}_{g_+} = O(1),$$

so

$$\begin{aligned} -2n \text{Vol}_{g_+}(X_\epsilon) \dot{\phantom{x}} &= \int_{X_\epsilon} (\dot{g}_{+\alpha\beta}^{\alpha\beta} - \dot{g}_{+\alpha}^{\alpha}{}_{,\beta}{}^{\beta}) dv_{g_+} + O(1) \\ &= \int_{\partial X_\epsilon} (\dot{g}_{+\alpha\beta}^{\alpha}{}_{,\beta} - \dot{g}_{+\alpha}^{\alpha}{}_{,\beta}{}^{\beta}) v_+^\beta d\sigma_+ + O(1), \end{aligned}$$

where  $\nu_+^\beta$  denotes the outward unit normal and  $d\sigma_+$  the induced volume density. The integral over  $x = \epsilon_0$  is  $O(1)$ , and on  $x = \epsilon$  we have  $\nu_+^\beta = -\epsilon\delta_0^\beta$ ,  $d\sigma_+ = \epsilon^{-n}dv_{g_\epsilon}$ . Also,  $\dot{g}_{+\alpha\beta}$  vanishes if either  $\alpha = 0$  or  $\beta = 0$ , and  $\dot{g}_{+ij} = x^{-2}\dot{g}_{ij}$ . An easy computation relating the connections of  $g_+$  and  $g_x$  shows that

$$\dot{g}_{+\alpha 0}^{\alpha} - \dot{g}_{+\alpha}^{\alpha},_0 = -\frac{1}{2}g^{ij}g^{kl}g'_{jl}\dot{g}_{ik} + x^{-1}g^{ij}\dot{g}_{ij} - (g^{ij}\dot{g}_{ij})',$$

which gives (3.4). □

Now  $\dot{L}$  occurs as the coefficient of  $\log \frac{1}{\epsilon}$  in the asymptotic expansion of the left hand side of (3.4). So we need to evaluate the  $\epsilon^{n-1} \log \frac{1}{\epsilon}$  coefficient in the expansion of the integral on the right-hand side. From  $dv_{g_x} = (\det g_x / \det g)^{1/2} dv_g$  and (3.2), it follows that the expansion of the volume form has no  $x^n \log x$  term, so does not contribute to this coefficient. Differentiating the volume form in  $t$ , one concludes that also the expansion of  $g^{ij}\dot{g}_{ij}$  has no  $x^n \log x$  term. Therefore the second and third terms in the integrand also do not contribute to the  $\epsilon^{n-1} \log \frac{1}{\epsilon}$  coefficient. The only contribution from the first term in the integrand comes from the log term in  $g'_{jl}$ . This gives  $2nc_n \dot{L} = \int_M \mathcal{O}_{ij} \dot{g}^{ij} dv_g$ , which combined with (3.3) gives Theorem 1.1.

## 4 Proof of Theorem 1.2

We consider natural tensor invariants of oriented  $n$ -dimensional Riemannian manifolds  $(M, g)$  with values in a subbundle  $\mathcal{V} \subset \otimes^k T^*M$  induced by a representation of  $\mathrm{SO}(n)$  on an invariant subspace  $V \subset \otimes^k(\mathbb{R}^n)^*$ ; see [E] for a discussion of natural tensors. The components of a natural tensor are expressible as linear combinations of partial contractions of the metric, the volume form, the Riemannian curvature tensor, and its covariant derivatives. We say that a natural tensor is irreducible if it is nonzero and if  $V$  is irreducible as an  $\mathrm{SO}(n)$ -module. We say that two natural tensors with values in subbundles  $\mathcal{V}_1, \mathcal{V}_2$  are equivalent if they correspond under an isomorphism  $\mathcal{V}_1 \cong \mathcal{V}_2$  induced by an  $\mathrm{SO}(n)$ -module isomorphism  $V_1 \cong V_2$  of the underlying subspaces.

The Ricci identity for commuting covariant derivatives does not preserve homogeneity degree, so the degree of a natural tensor as a polynomial in curvature and its derivatives is not well-defined. However, the space of natural tensors is filtered by degree and it does make sense to say that a natural tensor is of degree at least  $d$  in curvature for  $d \in \mathbb{N}$ . A natural tensor  $\mathcal{T}(g)$  is said to be conformally invariant of weight  $w$  if  $\mathcal{T}(\Omega^2 g) = \Omega^w \mathcal{T}(g)$  for  $0 < \Omega \in C^\infty(M)$ . The naturality of  $\mathcal{T}$  implies that if  $\phi$  is any local diffeomorphism, then  $\mathcal{T}(\phi^* g) = \phi^* \mathcal{T}(g)$ .

The idea of the proof of Theorem 1.2 is simple: linearizing a natural tensor at the usual metric on the sphere  $\mathbb{S}^n$  gives a linear differential operator on infinitesimal metrics, and the conformal invariance implies that this differential operator satisfies an invariance property under conformal motions. A known theorem classifies such

invariant differential operators and this classification in the linear case implies the classification up to quadratic and higher terms for natural tensors.

In more detail, let  $\mathcal{T}(g)$  be an irreducible natural tensor which is conformally invariant of weight  $w$ . Denote by  $g_0$  the usual metric on  $\mathbb{S}^n$ . Define a linear differential operator  $T$  from the bundle  $S_0^2 T^* \mathbb{S}^n$  of trace-free symmetric 2-tensors on  $\mathbb{S}^n$  to the bundle  $\mathcal{V}$  on  $\mathbb{S}^n$  by

$$T(h) = \frac{d}{dt} \mathcal{T}(g_0 + th)|_{t=0}$$

for  $h$  a section of  $S_0^2 T^* \mathbb{S}^n$ . We claim that if  $\phi : \mathbb{S}^n \rightarrow \mathbb{S}^n$  is a conformal motion of  $\mathbb{S}^n$  satisfying  $\phi^* g_0 = \Omega^2 g_0$  for  $0 < \Omega \in C^\infty(\mathbb{S}^n)$ , then

$$\Omega^w T(\phi^* h) = \phi^*(T(\Omega^2 \circ \phi^{-1} h)). \quad (4.1)$$

In fact,

$$\begin{aligned} \Omega^w \mathcal{T}(g_0 + t\phi^* h) &= \Omega^w \mathcal{T}(\Omega^{-2} \phi^* g_0 + t\phi^* h) = \Omega^w \mathcal{T}(\Omega^{-2} \phi^*(g_0 + t\Omega^2 \circ \phi^{-1} h)) \\ &= \mathcal{T}(\phi^*(g_0 + t\Omega^2 \circ \phi^{-1} h)) = \phi^* \mathcal{T}(g_0 + t\Omega^2 \circ \phi^{-1} h), \end{aligned}$$

from which the claim follows by differentiation.

Let  $G = O_e(n+1, 1)$  denote the identity component of the conformal group and  $P \subset G$  the isotropy group of a point on  $\mathbb{S}^n$ , so that  $\mathbb{S}^n = G/P$ . Then (4.1) states exactly that  $T$  is a  $G$ -equivariant map between sections of the homogeneous bundles  $S_0^2 T^* \mathbb{S}^n(2)$  and  $\mathcal{V}(w)$  on  $G/P$ , where the number in parentheses indicates the conformal weight of the homogeneous bundle. Such invariant differential operators between any irreducible homogeneous bundles on  $\mathbb{S}^n$  have been completely classified; see (1.4) of [BC]. The classification in [BC] is formulated in terms of the homomorphisms of the generalized Verma modules dual to the homomorphisms of the modules of jets of sections of the homogeneous bundles induced by the differential operators. See [BE], [ES] and references cited there for elaboration and interpretation of this classification in the context of conformal geometry.

For our purposes it is sufficient to know all the invariant operators with domain  $S_0^2 T^* \mathbb{S}^n(2)$  and range any irreducible bundle. The bundle  $S_0^2 T^* \mathbb{S}^n(2)$  has regular integral infinitesimal character so fits into a generalized Bernstein–Gelfand–Gelfand complex of invariant operators (the so called deformation complex – see [GG] for a direct construction of this complex on a general conformally flat manifold). In odd dimensions, up to scale and equivalence, there is precisely one invariant operator acting on  $S_0^2 T^* \mathbb{S}^n(2)$ : the linearized Cotton tensor in dimension 3 and the linearized Weyl tensor in higher dimensions. In even dimensions there are further operators. For  $n \geq 6$  there is one more operator acting on  $S_0^2 T^* \mathbb{S}^n(2)$ , which must be the linearized obstruction tensor since this is a nonzero operator acting between the appropriate bundles. The case  $n = 4$  is exceptional because  $S_0^2 T^* \mathbb{S}^n(2)$  occurs at the edge of the middle diamond of the Hasse diagram, and there are three invariant operators acting on  $S_0^2 T^* \mathbb{S}^n(2)$ , which can be identified with the linearized self-dual and anti-self-dual Weyl tensors and the linearized Bach tensor.

Theorem 1.2 is an immediate consequence. The linearization of a conformally invariant irreducible natural tensor is equivalent to a multiple of one of the invariant operators given by the Boe–Collingwood classification. By inspection, each such operator is the linearization of one of the conformally invariant natural tensors listed in the statement of the Theorem. But if two conformally invariant natural tensors have the same linearization on the sphere, their difference must be of degree at least 2 in curvature.

## References

- [Al] S. Alexakis, in preparation.
- [An] M. Anderson,  $L^2$  curvature and volume renormalization for AHE metrics on 4-manifolds, *Math. Res. Lett.* 8 (2001), 171–188.
- [B] T. Branson, Sharp inequalities, the functional determinant, and the complementary series, *Trans. Amer. Math. Soc.* 347 (1995), 3671–3742.
- [BC] B. D. Boe and D. H. Collingwood, A comparison theory for the structure of induced representations, *J. Algebra* 94 (1985), 511–545.
- [BE] R.J. Baston and M.G. Eastwood, Invariant operators, in: *Twistors in Mathematics and Physics*, London Math. Soc. Lecture Note Ser. 156, Cambridge University Press, Cambridge 1990, 129–163.
- [HSS] S. de Haro, K. Skenderis and S.N. Solodukhin, Holographic reconstruction of space-time and renormalization in the AdS/CFT correspondence, *Comm. Math. Phys.* 217 (2001), 594–622, hep-th/0002230.
- [ES] M. Eastwood and J. Slovák, Semiholonomic Verma modules, *J. Algebra* 197 (1997), 424–448.
- [E] D. Epstein, Natural tensors on Riemannian manifolds, *J. Differential Geom.* 10 (1975), 631–645.
- [FG1] C. Fefferman and C.R. Graham, Conformal invariants, in: *Élie Cartan et les mathématiques d’aujourd’hui* (Lyon 1984), *Astérisque* (1985), Numéro hors-série, 95–116.
- [FG2] C. Fefferman and C.R. Graham,  $Q$ -curvature and Poincaré metrics, *Math. Res. Lett.* 9 (2002), 139–151.
- [FH] C. Fefferman and K. Hirachi, Ambient metric construction of  $Q$ -curvature in conformal and CR geometries, *Math. Res. Lett.* 10 (2003), 819–832.
- [GG] J. Gasqui and H. Goldschmidt, *Déformations Infinitésimales des Structures Conformes Plates*, Progr. Math. 52, Birkhäuser, Boston 1984.
- [G] C. R. Graham, Volume and area renormalizations for conformally compact Einstein metrics, *Rend. Circ. Mat. Palermo, Ser. II, Suppl.* 63 (2000), 31–42.
- [GJMS] C. R. Graham, R. Jenne, L. J. Mason, and G. A. J. Sparling, Conformally invariant powers of the Laplacian. I. Existence. *J. London Math. Soc.* (2) 46 (1992), 557–565.

- [GL] C. R. Graham and J. M. Lee, Einstein metrics with prescribed conformal infinity on the ball, *Adv. Math.* 87 (1991), 186–225.
- [GZ] C.R. Graham and M. Zworski, Scattering matrix in conformal geometry, *Invent. Math.* 152 (2003), 89–118, [math.DG/0109089](#).





# AdS/CFT correspondence and geometry

*Ioannis Papadimitriou and Kostas Skenderis*

*Institute for Theoretical Physics, University of Amsterdam  
Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands  
email: [ipapadim@science.uva.nl](mailto:ipapadim@science.uva.nl), [skenderi@science.uva.nl](mailto:skenderi@science.uva.nl)*

**Abstract.** In the first part of this paper we provide a short introduction to the AdS/CFT correspondence and to holographic renormalization. We discuss how QFT correlation functions, Ward identities and anomalies are encoded in the bulk geometry. In the second part we develop a Hamiltonian approach to the method of holographic renormalization, with the radial coordinate playing the role of time. In this approach regularized correlation functions are related to canonical momenta and the near-boundary expansions of the standard approach are replaced by covariant expansions where the various terms are organized according to their dilatation weight. This leads to universal expressions for counterterms and one-point functions (in the presence of sources) that are valid in all dimensions. The new approach combines optimally elements from all previous methods and supersedes them in efficiency.

## Contents

|   |                                                               |    |
|---|---------------------------------------------------------------|----|
| 1 | Introduction . . . . .                                        | 73 |
| 2 | QFT data . . . . .                                            | 75 |
| 3 | AdS/CFT correspondence . . . . .                              | 79 |
| 4 | Hamiltonian approach to holographic renormalization . . . . . | 82 |
|   | 4.1 Pure gravity case . . . . .                               | 90 |
|   | 4.2 Gravity coupled to scalars . . . . .                      | 95 |
| 5 | Conclusions . . . . .                                         | 98 |

## 1 Introduction

The AdS/CFT correspondence [21], [17], [31] (for reviews see [1], [8]) relates string theory on (locally asymptotically) AdS spacetimes (times a compact space) with a quantum field theory (QFT) “residing” on the conformal boundary of the bulk spacetime.<sup>1</sup> In a specific limit, which is a strong coupling limit of the boundary theory, the

---

<sup>1</sup>In the first examples discussed in the literature, the bulk spacetime was exactly AdS (times a compact space) and the dual theory was a conformal field theory (CFT). This motivated the name “AdS/CFT correspondence”. Our discussion is applicable under the more general circumstances mentioned above.

bulk theory reduces to classical gravity coupled to certain matter fields. In this limit QFT data is encoded in classical geometry. The aim of this contribution is to discuss how to extract this data from the bulk geometry.

In the first part of this paper we give a brief review of the AdS/CFT correspondence and of holographic renormalization for non-experts. We start our discussion from the QFT side by reviewing what QFT data we would like to obtain from gravity. This data consists of correlation functions of gauge invariant operators and of symmetry relations. Symmetries of the QFT action imply relations among correlation functions, the so-called Ward identities. Such relations are “kinematical” and can be established without the need to actually compute the correlation functions. Sometimes, however, quantum effects imply that some of the classical symmetries are broken at the quantum level: the corresponding Ward identities are anomalous. The symmetries of the quantum theory are thus encoded in the corresponding Ward identities and anomalies.

The next task is to describe how to obtain correlation functions, Ward identities and anomalies using the AdS/CFT correspondence. We outline the prescription of [17], [31] for the computation of correlation functions and discuss how to deal with the infinities arising in such computations. This is dealt with via the formalism of holographic renormalization [18], [10], [5] (for a review see [26]; for related work see [3], [20], a more complete list of references can be found in the review). This formalism automatically incorporates the “kinematical” constraints, i.e. the Ward identities and their anomalies, and identifies the part of the geometry where the “dynamical” information, i.e. the correlation functions, is encoded. A central role in this method is played by the fact that one can perturbatively work out the asymptotics of all bulk fields to sufficiently high order using the radial distance from the boundary of AdS as a small parameter<sup>2</sup> [12] (for relevant math reviews see [15], [2]). Correlation functions are encoded in specific coefficients in the asymptotic expansion of the bulk fields and Ward identities and anomalies originate from certain relations that these coefficients satisfy.

This method, even though complete, is not very efficient as we discuss later. The last part of the paper is devoted to developing a “Hamiltonian” version of the method, where the radial coordinate plays the role of time. In this approach the focus is shifted from the on-shell supergravity action to the canonical momenta of the bulk fields. The latter are associated with (regularized) correlation functions of gauge invariant operators [9]. To obtain renormalized correlation functions we need to subtract the infinities. This was done in the standard approach via the near-boundary analysis. In the new approach we use instead the fact that there is a well defined dilatation operator. This allows us to develop a covariant expansion of the asymptotic solution where the various terms are organized according to their dilatation eigenvalue. This leads to a faster algorithm for determining counterterms and correlation functions.<sup>3</sup>

---

<sup>2</sup>From the point of view of the dual field theory we expand around a UV fixed point, the small parameter being the inverse energy.

<sup>3</sup>A different Hamiltonian approach to renormalization using the Hamilton–Jacobi equation [9] was developed in [22], see also [14] and references therein.

In particular, we obtain universal recursion relations for the asymptotic solutions and counterterms that are valid in all dimensions.

This paper is organized as follows. In the next section we discuss the QFT data that enters in the discussion of the AdS/CFT correspondence. The discussion is illustrated by a number of simple examples and assumes only a general familiarity with QFT. In Section 3 we present a brief review of the AdS/CFT correspondence and of holographic renormalization. More details can be found in the reviews listed above. Sections 2 and 3 are aimed at non-experts that want to get a flavor of the ideas and techniques involved in the AdS/CFT correspondence. Experts can safely move directly to Section 4 where the new Hamiltonian formulation is discussed systematically. This section is self-contained and can be read independently of the previous sections.

Throughout this paper we work with Euclidean signature. All results, however, can be straightforwardly continued to any other signature.

## 2 QFT data

We discuss in this section the quantum field theory data that we would like to extract from the bulk geometry. We would say that a QFT is solved if we determine all correlation functions of all gauge invariant operators. The set of gauge invariant operators depends on the theory under consideration. Examples of such operators are the stress energy tensor  $T_{ij}$ , currents  $J^i$  associated with global symmetries and scalar operators  $O$ . As mentioned in the introduction, when the bulk spacetime is exactly AdS (times a compact space) the dual quantum field theory is a conformal field theory (CFT). The discussion in this section will refer to CFTs, but all considerations have a straightforward generalization to QFTs that can be viewed either as deformations of the CFTs by relevant or marginal operators or to CFTs with spontaneously broken conformal invariance.

The examples of AdS/CFT correspondence involve specific CFTs, the most studied case being the maximally supersymmetric gauge theory in four dimensions, the  $\mathcal{N} = 4$  SYM theory. The discussion below is focused on general properties that do not depend on the details of the specific CFT. Given a (perturbative) CFT specified by set of fields  $\varphi^A$  one can work out the set of gauge invariant composite operators  $\mathcal{O}(\varphi^A)$ . Their correlation functions can then be computed in perturbation theory. To give an elementary example of a CFT correlator consider a scalar operator  $O_\Delta$  of conformal weight  $\Delta$ . In this case the form of the 2-point function is fixed by conformal invariance,

$$\langle O_\Delta(x) O_\Delta(0) \rangle = \frac{c(g, \Delta)}{x^{2\Delta}}, \quad (1)$$

where  $c(g, \Delta)$  is a constant that depends on the coupling constant of the theory  $g$  and the conformal dimension  $\Delta$  of the operator. One may set it to one by a choice of normalization of  $O_\Delta$  but we shall not do so. Our objective is to understand how to

extract this and more complicated higher point functions (that are not determined by symmetries) from the bulk geometry.

In general, symmetries of the classical action imply relations among correlation functions, the so-called Ward identities. To give an example: Poincaré invariance of the classical action implies (classically) that the stress energy tensor is conserved,

$$\partial^i T_{ij} = 0. \quad (2)$$

At the quantum level this implies relations among certain correlation functions. For instance,

$$\partial_x^i \langle T_{ij}(x) O(y) O(z) \rangle = \partial_j \delta(x - y) \langle O(x) O(z) \rangle + \partial_j \delta(x - z) \langle O(y) O(x) \rangle. \quad (3)$$

Some of the classical symmetries, however, are broken by quantum effects. For example, the stress energy tensor of a field theory that is classically conformally invariant is traceless, but quantum effects may break this symmetry

$$T_i^i = 0 \text{ classical}, \quad \langle T_i^i \rangle = \mathcal{A} \text{ quantum}. \quad (4)$$

Since  $T_i^i$  generates scale transformations, the conformal anomaly captures the fact that the correlators, even though they are CFT correlators, are not scale invariant,

$$\mu \frac{\delta}{\delta \mu} \langle O_1(x_1) \dots O_n(x_n) \rangle = A \delta(x_1, \dots, x_n) \quad (5)$$

where  $\delta(x_1, \dots, x_n) = \delta(x_1 - x_2) \delta(x_2 - x_3) \dots \delta(x_{n-1} - x_n)$ , and  $A$  is related to  $\mathcal{A}$  in a way we specify below. Notice that the violation of conformal invariance is a contact term. In a general QFT (i.e. not conformal) (5) is replaced by the beta function equation.

To understand how an anomaly can arise consider the 2-point function in (1). The form of this correlator (for  $x^2 \neq 0$ ) is completely fixed (up to normalization) by conformal invariance. Depending on the conformal dimension, however, the correlator may suffer from short distance singularities. Consider the case  $\Delta \sim d/2 + k$ , where  $d$  is the spacetime dimension and  $k$  is an integer. As  $x^2 \rightarrow 0$  the correlator behaves as

$$\frac{1}{x^{2\Delta}} \sim \frac{1}{d + 2(k - \Delta)} \frac{\Gamma(d/2)}{2^{2k} k! \Gamma(d/2 + k)} S^{d-1} \square^k \delta^{(d)}(x) \quad (6)$$

where  $S^{d-1} = 2\pi^{d/2} / \Gamma(d/2)$  is the volume of the unit  $(d-1)$ -sphere and  $\square = \delta^{ij} \partial_i \partial_j$ . We thus find that there is a pole at  $\Delta = d/2 + k$ . To produce a well-defined distribution we use dimensional regularization and subtract the pole. For concreteness we consider the case  $k = 0$  (all other cases follow by differentiation w.r.t. to  $x$ , see [25]). Minimal subtraction yields [23], [25]

$$\begin{aligned} \langle O_{d/2}(x) O_{d/2}(0) \rangle_R &= c(g, d/2) \lim_{\Delta \rightarrow d/2} \left( \frac{1}{x^{2\Delta}} - \frac{\mu^{2\Delta-d}}{d - 2\Delta} S^{d-1} \delta^{(d)}(x) \right) \\ &= -c(g, d/2) \frac{1}{2(d-2)} \square \frac{1}{(x^2)^{\frac{1}{2}d-1}} \left( \log \mu^2 x^2 + \frac{2}{d-2} \right), \end{aligned} \quad (7)$$

where the subscript  $R$  indicates that this is a renormalized correlator. The scale  $\mu$  is introduced, as usual in dimensional regularization, on dimensional grounds. The renormalized correlator agrees with the bare one away from coincident points but is also well-defined at  $x^2 = 0$ . Let us now consider the scale dependence of the renormalized correlator

$$\mu \frac{\partial}{\partial \mu} \langle O_{d/2}(x) O_{d/2}(0) \rangle_R = S^{d-1} c(g, d/2) \delta^{(d)}(x), \quad (8)$$

where we used  $\square(x^2)^{-d/2+1} = -(d-2)S^{d-1}\delta^{(d)}(x)$ . Thus the renormalized correlation function exhibits a violation of scale invariance. We shall soon connect this to the violation of the tracelessness condition of the stress energy tensor.

Recall that correlation function of composite operators may be computed by introducing sources that couple to them. The generating functional of correlation functions then has the following path integral representation

$$Z[g_{(0)}, \phi_{(0)}] = \int [D\varphi^A] \exp \left( - \int d^d x \sqrt{g_{(0)}} [\mathcal{L}_{\text{CFT}}(\varphi^A; g_{(0)}) + \phi_{(0)} O(\varphi^A)] \right) \quad (9)$$

where  $\varphi^A$  represents collectively all fields of the theory,  $g_{(0)}$  is a background metric (which serves as a source for the stress energy tensor),  $\mathcal{L}_{\text{CFT}}$  is the Lagrangian density for the CFT and  $\phi_{(0)}$  is a source for the operator  $O$ . Correlation functions can now be computed by differentiating w.r.t. sources and then setting the sources to zero. For instance, the connected two-point function of  $O$  on flat spacetime ( $g_{(0)ij} = \delta_{ij}$ ) is given by

$$\langle O(x) O(0) \rangle = \frac{\delta^2 W}{\delta \phi_{(0)}(x) \delta \phi_{(0)}(0)} \Big|_{\phi_{(0)}=0} \quad (10)$$

where  $W = \log Z$  is the generating functional of connected correlators. Given a Lagrangian density  $\mathcal{L}_{\text{QFT}}$  one could thus compute the correlation functions of  $O$  by first (perturbatively) computing  $Z[g_{(0)}, \phi_{(0)}]$ . Such computations however are plagued by infinities and to make sense of them one needs to renormalize the theory. To subtract the divergences one may add counterterms to the action. If the counterterms break a classical symmetry, then this symmetry is anomalous.

A slightly different route is to first compute in general the one-point functions in the presence of sources,

$$\langle T_{ij}(x) \rangle_s = - \frac{2}{\sqrt{g_{(0)}(x)}} \frac{\delta W[g_{(0)}, \phi_{(0)}]}{\delta g_{(0)}^{ij}(x)}, \quad \langle O(x) \rangle_s = - \frac{1}{\sqrt{g_{(0)}(x)}} \frac{\delta W[g_{(0)}, \phi_{(0)}]}{\delta \phi_{(0)}(x)} \quad (11)$$

where the subscript  $s$  in the correlation functions indicate that the sources are non-zero. Correlation functions are then computed by further differentiating w.r.t. sources and setting the sources to zero. This reformulation will be proved useful later when we show how to compute correlation functions holographically (i.e. using the AdS/CFT correspondence). Another advantage is that one can express compactly many Ward

identities. For instance, invariance of  $Z$  under diffeomorphisms

$$\delta g_{(0)}^{ij} = -(\nabla^i \xi^j + \nabla^j \xi^i), \quad \delta \phi_{(0)} = \xi^j \nabla_j \phi_{(0)} \quad (12)$$

implies

$$\nabla^i \langle T_{ij}(x) \rangle_s = -\langle O(x) \rangle_s \nabla_j \phi_{(0)}(x). \quad (13)$$

Differentiating now twice w.r.t.  $\phi_{(0)}$  and then setting  $\phi_{(0)} = 0$ ,  $g_{(0)ij} = \delta_{ij}$  leads to (3).

Using the fact that the trace of the stress energy tensor is the generator of conformal transformations we arrive at [25]

$$\int d^d x \sqrt{g_{(0)}} g_{(0)}^{ij} \langle T_{ij} \rangle = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int \prod_{i=1}^k (d^d x_i \sqrt{g_{(0)}} J(x_i)) \mu \frac{\partial}{\partial \mu} \langle \mathcal{O}(x_1) \dots \mathcal{O}(x_k) \rangle \quad (14)$$

where  $J$  denotes all sources and  $\mathcal{O}$  the corresponding operators. In our case,  $J = \{\phi_{(0)}, g_{(0)ij}\}$  and  $\mathcal{O} = \{O, T_{ij}\}$ . Clearly, the expectation value of the stress energy tensor is non-vanishing if the scale derivative of the correlator is non-vanishing. In particular, we have seen in (8) that the scale derivative of the 2-point function of an operator of dimension  $d/2$  yields a delta function. Inserting this in (14) we obtain

$$\langle T_i^i \rangle = \frac{1}{2} S^{d-1} c(g, d/2) \phi_{(0)}^2. \quad (15)$$

So, in this case,  $\mathcal{A} = A \phi_{(0)}^2/2$  and  $A = S^{d-1} c(g, d/2)$ . This result generalizes to all operators of dimension  $\Delta = d/2 + k$  with result [25]

$$\langle T_i^i \rangle = \frac{1}{2} c_k \phi_{(0)} \square^k \phi_{(0)}, \quad c_k = \frac{\pi^{d/2}}{2^{2k-1} \Gamma(k+1) \Gamma(k+d/2)} c(g, \Delta). \quad (16)$$

These considerations were valid for flat spacetime. When the background is curved, the results generalize to

$$\langle T_i^i \rangle = \frac{1}{2} c_k \phi_{(0)} P_k \phi_{(0)} + \left( a E + \sum_i c_i W^i \right) + \nabla_i J^i. \quad (17)$$

$P_k$  is equal to  $\square^k$  when the background is flat and transforms covariantly under Weyl transforms  $g_{(0)} \rightarrow g_{(0)} e^{2\sigma}$

$$P_k \rightarrow e^{-(d/2+k)\sigma} P_k e^{(d/2-k)\sigma}. \quad (18)$$

For instance, for  $k = 1$

$$P_1 = \square + \frac{d-2}{4(d-1)} R. \quad (19)$$

The two terms inside the parenthesis in (17) are purely gravitational and are present only when  $d$  is even.  $E$  is the Euler density,  $W^i$  is a basis of Weyl invariants of dimension  $d$  and  $a$  and  $c_i$  are numerical constants that depend on the field content of

the theory. For instance, in  $d = 4$  there is one Weyl invariant (the square of the Weyl tensor), in  $d = 6$  there are three such tensors, etc. The last term in (17) is scheme dependent, i.e. it can be modified by local finite counterterm terms in the action. In general there may be additional terms in (17) that depend on higher powers of the sources  $\phi_{(0)}$ . These would be related to singularities in higher-point functions. The structure of (17) is dictated by the fact that the integrated conformal anomaly is itself conformally invariant [6], [11].

The AdS/CFT duality implies that all this data is encoded in the geometry. We discuss in the next section how to recover them from the bulk geometry.

### 3 AdS/CFT correspondence

The AdS/CFT correspondence states that there is an exact equivalence between string theory on (locally) asymptotically AdS (AAdS) spacetimes (times a compact space) and a quantum field theory that “resides” on the conformal boundary of the AAdS spacetime. In the regime where the one description is perturbative the other one is strongly coupled. We will work in the regime where the gravitational description is valid and we will describe how to obtain the QFT data described in the previous section.

The basic AdS/CFT dictionary is as follows:

- (1) Gauge invariant operators of the boundary theory are in one-to-one correspondence with bulk fields. For example, the bulk metric corresponds to the stress energy tensor of the boundary theory.
- (2) The leading boundary behavior of the bulk field is identified with the source of the dual operator.
- (3) The string partition function (which is a functional of the fields parameterizing the boundary behavior of the bulk fields) is identified with the generating functional of QFT correlation functions.

At low energies and to leading order the AdS/CFT prescription reads

$$S_{\text{on-shell}}[f_{(0)}] = -W[f_{(0)}] \quad (20)$$

where  $S_{\text{on-shell}}[f_{(0)}]$  is the on-shell value of the supergravity action,  $f_{(0)}$  denotes collectively all fields parameterizing the boundary values of bulk fields and  $W$  is the generating functional of connected graphs (see the discussion below (9)). It follows that one can compute correlators of the (strongly coupled) QFT gravitationally by first evaluating the on-shell value of the supergravity action and then differentiating w.r.t. the boundary values, e.g.

$$\langle O(x) O(0) \rangle = - \frac{\delta^2 S_{\text{on-shell}}}{\delta \phi_{(0)}(x) \delta \phi_{(0)}(0)} \Big|_{\phi_{(0)}=0}. \quad (21)$$



A naive use of these formulas however yields infinite answers – the on-shell value of the action is infinite due to the infinite volume of the AAdS spacetime. The goal of holographic renormalization is to make these formulas well-defined.

The general form of the bulk action is

$$S = \int d^{d+1}x \sqrt{g} \left[ -\frac{1}{2\kappa^2} R + \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + V(\Phi) + \dots \right] \quad (22)$$

where  $\kappa^2 = 8\pi G_{d+1}$  ( $G_{d+1}$  is Newton's constant) and the dots indicate contribution of additional fields such as gauge fields, fermions, antisymmetric tensors. The analysis below generalizes straightforwardly to include such fields (but it becomes a lot more tedious). Restricting to the gravity-scalar sector means that we only study correlation functions of the stress energy tensor and a scalar operator. The potential has the form

$$V(\Phi) = \frac{\Lambda}{\kappa^2} + \frac{1}{2} m^2 \Phi^2 + g \Phi^3 + \dots \quad (23)$$

where  $\Lambda$  is the cosmological constant and the mass  $m^2$  of the scalar field is related to the dimension  $\Delta$  of the dual operator by  $m^2 = (\Delta - d)\Delta$ . The bulk field equations are given by

$$G_{\mu\nu} = \kappa^2 \tilde{T}_{\mu\nu}(\Phi), \quad \square_g \Phi = \partial V / \partial \Phi \quad (24)$$

where  $G_{\mu\nu}$  is the Einstein tensor,  $\square_g \Phi = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu \Phi)$  and  $\tilde{T}_{\mu\nu}(\Phi)$  is the stress energy tensor associated with the scalar field  $\Phi$  (see (34)).

The method of holographic renormalization now consists of the following steps (a more detailed discussion can be found in [26]).

**1. Asymptotic solutions.** In the first step one works out the most general asymptotic solutions with given Dirichlet data

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{ij}(x, \rho) dx^i dx^j, \quad (25)$$

$$\Phi(x, \rho) = \rho^{(d-\Delta)/2} \phi(x, \rho) \quad (26)$$

where<sup>4</sup>

$$g_{ij}(x, \rho) = g_{(0)ij} + \rho g_{(2)ij} + \dots + \rho^{d/2} (g_{(d)ij} + \log \rho h_{(d)ij}) + \dots, \quad (27)$$

$$\phi(x, \rho) = \phi_{(0)} + \rho \phi_{(2)} + \dots + \rho^{\Delta-d/2} (\phi_{(2\Delta-d)} + \log \rho \psi_{(2\Delta-d)}) + \dots. \quad (28)$$

In this expansion,  $g_{(0)ij}$  and  $\phi_{(0)}$  are identified with the QFT sources that couple to the dual operators, as discussed in the previous section.

Inserting these expansions in the bulk field equations (24) one finds that all coefficients but  $\phi_{(2\Delta-d)}$  and the traceless transverse part of  $g_{(d)ij}$  are locally determined by  $g_{(0)ij}$  and  $\phi_{(0)}$  [12], [18], [10] (see the appendices of [10] for explicit expressions of the

<sup>4</sup>In general, the expansions may involve half integral powers of  $\rho$ . In such cases it is more natural to use a new radial coordinate  $r$ , where  $\rho = r^2$ .

coefficients). The part of  $g_{(d)ij}$  that is determined, i.e.  $\nabla^i g_{(d)ij}$  and  $\text{Tr } g_{(d)}$ , encodes Ward identities and anomalies, as we discuss below. We will call  $g_{(d)ij}$  and  $\phi_{(2\Delta-d)}$  the response functions. The logarithmic terms appear only in special cases:  $h_{(d)}$  only in even dimensions and  $\psi_{(2\Delta-d)}$  only when  $\Delta - d/2$  is an integer. Both of them are directly related to the conformal anomalies discussed in the previous section:  $h_{(d)}$  is the metric variation of the gravitational part of the conformal anomaly and  $\psi_{(2\Delta-d)}$  is the variation w.r.t.  $\phi_{(0)}$  of the matter part of the conformal anomaly [10].

**2. On-shell divergences.** Having obtained the asymptotic solutions we now obtain the most general divergences of the on-shell action

$$S_{\text{reg}}[g_{(0)}, \phi_{(0)}; \epsilon] = \int d^d x \sqrt{g_{(0)}} \left( \sum_{\nu} a_{(\nu)} \epsilon^{-\nu} - \log \epsilon a_{(d)} \right) + \mathcal{O}(\epsilon) \quad (29)$$

where  $\epsilon$  is a cut-off in the radial coordinate,  $\rho \geq \epsilon$ . It turns out all coefficients  $a_{(\nu)}$  depend only on  $g_{(0)}$  and  $\phi_{(0)}$  but not on the undetermined coefficients  $g_{(d)}$  and  $\phi_{(2\Delta-d)}$ . The coefficient  $a_{(d)}$  is equal to the conformal anomaly of the dual CFT [18].

**3. Counterterms and renormalized action.** To obtain a well-defined on-shell action we should subtract the infinities and then remove the regulator. To do this we first express the divergent terms found in the previous step in terms of induced fields at the hypersurface  $\rho = \epsilon$ . This entails inverting the asymptotic series obtained in the first step and inserting it in the divergent terms obtained in the second step. This is one of the most laborious steps of the procedure. The end result is the counterterm action,  $S_{\text{ct}}$ . The renormalized action is defined by

$$S_{\text{ren}} = \lim_{\epsilon \rightarrow 0} S_{\text{sub}}, \quad S_{\text{sub}} = S_{\text{reg}} + S_{\text{ct}}. \quad (30)$$

**4. 1-point functions in the presence of source.** We can now differentiate the renormalized action to obtain the 1-point function in the presence of sources [10]:

$$\begin{aligned} \langle T_{ij}(x) \rangle_s &\equiv \frac{2}{\sqrt{g_{(0)}(x)}} \frac{\delta S_{\text{ren}}}{\delta g_{(0)}^{ij}(x)} = \lim_{\epsilon \rightarrow 0} \left( \frac{1}{\epsilon^{d/2-1}} \frac{2}{\sqrt{\gamma(x, \epsilon)}} \frac{\delta S_{\text{sub}}}{\delta \gamma^{ij}(x, \epsilon)} \right) \\ &= \frac{d}{2\kappa^2} g_{(d)ij} + X[g_{(0)}, \phi_{(0)}], \\ \langle O(x) \rangle_s &\equiv \frac{1}{\sqrt{g_{(0)}(x)}} \frac{\delta S_{\text{ren}}}{\delta \phi_{(0)}(x)} = \lim_{\epsilon \rightarrow 0} \left( \frac{1}{\epsilon^{\Delta/2}} \frac{1}{\sqrt{\gamma(x, \epsilon)}} \frac{\delta S_{\text{sub}}}{\delta \Phi(x, \epsilon)} \right) \\ &= (d - 2\Delta) \phi_{(2\Delta-d)} + Y[g_{(0)}, \phi_{(0)}] \end{aligned} \quad (31)$$

where  $X[g_{(0)}, \phi_{(0)}]$  and  $Y[g_{(0)}, \phi_{(0)}]$  are (known) local expressions that depend on sources. The first equality is a definition. In the second equality we expressed the 1-point function as a limit of the regulated 1-point function. The regulated 1-point function can be computed in all generality and the limit can be explicitly taken. This is a straightforward but rather tedious computation. The result is the one shown above. We thus find that the correlation functions depend on the coefficient that the asymptotic analysis left undetermined. As discussed above, the near boundary analysis does

determine the divergence and trace of  $g_{(d)ij}$ . This means that the divergence and trace of  $\langle T_{ij}(x) \rangle_s$  can be determined. This yields the Ward identities, including anomalies, that we discussed in the previous section. The relations (31) imply that the pairs  $(g_{(0)}, g_{(d)})$  and  $(\phi_{(0)}, \phi_{(2\Delta-d)})$  are conjugate pairs.

**5. Correlation functions.** To obtain higher point functions we should further differentiate (31) w.r.t. the sources. The expressions  $X[g_{(0)}, \phi_{(0)}]$  and  $Y[g_{(0)}, \phi_{(0)}]$  lead to only local contributions. The (non-local)  $n$ -point function is thus encoded in the dependence of  $g_{(d)}$  and  $\phi_{(2\Delta-d)}$  on the sources. We thus reach the conclusion:

*The theory is solved if we determine the response functions in terms of the sources.*

To obtain such a relation we need a regular exact (as opposed to asymptotic) solution of the bulk equations with the boundary conditions specified by the sources. In the absence of more powerful methods one can proceed perturbatively. One can determine the response functions to linear order by solving the bulk field equations linearized around a background solution [5]. (The background solution specifies the vacuum of the dual QFT, see Section 6.1 of [26].) Higher-point functions can be computed by solving the bulk equations perturbatively in a bulk coupling constant. Examples have been discussed in [26], [4].

The procedure described here is general and can be carried out in all cases. The steps however appear to have certain redundancy. In step 1 and 2 the asymptotic solution and divergences are obtained in terms of the Dirichlet data. In order to obtain the counterterms however one should invert the asymptotic series. Then the 1-point functions are obtained in terms of the induced fields at  $\rho = \epsilon$  and the asymptotic solution is used again to obtain the final expression for 1-point functions. Clearly, it would be desirable to avoid having to go back and forth from asymptotic data to covariant fields. A related issue is the following. In step 2 we mentioned that the divergences depend only on the sources but not the response functions. This followed from an explicit computation. It would be more satisfying to make this manifest. We discuss in the next section an approach that removes these drawbacks. A related work that also leads to simplifications can be found in [22].

## 4 Hamiltonian approach to holographic renormalization

Let  $\bar{M}$  be a conformally compact, Riemannian  $(d+1)$ -manifold,  $M$  its interior and  $\partial M$  its boundary. We will consider the following action for the Riemannian metric  $g_{\mu\nu}$  on  $\bar{M}$

$$S_{\text{gr}}[g] = -\frac{1}{2\kappa^2} \left[ \int_M d^{d+1}x \sqrt{g} R + \int_{\partial M} d^d x \sqrt{\gamma} 2K \right], \quad (32)$$

where  $\kappa^2 = 8\pi G_{d+1}$ ,  $\gamma$  is the induced metric on  $\partial M$  and  $K$  is the trace of the extrinsic curvature of the boundary. This is the standard Einstein–Hilbert action with

the Gibbons–Hawking boundary term which ensures that the variational problem is well-defined. The overall sign is chosen so that the action is positive definite when evaluated on a classical solution in the vicinity of (Euclidean) AdS.<sup>5</sup> To allow for matter we add

$$S_m = \int_M d^{d+1}x \sqrt{g} \mathcal{L}_m \quad (33)$$

to the gravitational action, where  $\mathcal{L}_m$  is a generic matter field Lagrangian density. The stress tensor is then defined in the standard fashion by

$$\delta_g S_m \equiv \frac{1}{2} \int_M d^{d+1}x \sqrt{g} \tilde{T}_{\mu\nu} \delta g^{\mu\nu}. \quad (34)$$

The Euler–Lagrange equations of the total action  $S = S_{gr} + S_m$  are Einstein’s equations

$$G_{\mu\nu} = \kappa^2 \tilde{T}_{\mu\nu} \quad (35)$$

and the matter field equations.

Our method of holographic renormalization makes use of the ADM formalism and the Gauss–Codacci equations which we will now briefly review. The standard ADM formalism (see, for instance, [30]) for a pseudo-Riemannian manifold relies on the existence of a *global* time function  $t$  which is used to foliate spacetime into diffeomorphic hypersurfaces of constant  $t$ . For a generic Riemannian manifold, however, there is no natural choice of time as all coordinates are equivalent. Nevertheless, for a Riemannian manifold with boundary one can use the coordinate ‘normal’ to the boundary as a global ‘time’ coordinate and, hence, foliate the manifold into hypersurfaces diffeomorphic to the boundary. For asymptotically (Euclidean) AdS manifolds this can always be done at least in a neighborhood of the boundary [12] (see also the recent review [2] and references therein). The question of if and where this ‘radial’ coordinate emanating from the boundary ceases to be well-defined depends on the topology of the space and will not be addressed here.

Let  $r$  be the ‘radial’ coordinate emanating from the boundary of a Riemannian manifold with boundary  $(M, g_{\mu\nu})$  in the way described above and consider the hypersurfaces  $\Sigma_r$  defined by  $r(x) = \text{constant}$ . The unit normal to  $\Sigma_r$ , pointing in the direction of increasing  $r$ , is given by  $n^\mu = \frac{1}{\|dr\|_g} g^{\mu\nu} \frac{\partial r}{\partial x^\nu} \big|_\Sigma$ . This allows one to express the induced metric on the hypersurfaces in a coordinate independent fashion as<sup>6</sup>  $\hat{\gamma}_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$ . The metric on  $M$  can then be decomposed as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \hat{\gamma}_{\mu\nu} d\hat{x}^\mu d\hat{x}^\nu + 2N_\mu d\hat{x}^\mu dr + (N^2 + N_\mu N^\mu) dr^2 \quad (36)$$

where  $N$  and  $N^\mu$  are respectively the *lapse function* and the *shift function*. They correspond to non-dynamical degrees of freedom which we will ‘gauge-fix’ shortly.

<sup>5</sup>Our convention for the Riemann tensor is  $R^\mu{}_{\rho\nu\sigma} = \partial_\nu \Gamma^\mu_{\rho\sigma} + \Gamma^\mu_{\lambda\nu} \Gamma^\lambda_{\rho\sigma} - (v \leftrightarrow \sigma)$ . This differs by an overall sign from the conventions used in [18], [10].

<sup>6</sup>We use a hat to denote tensors that are purely transverse to the unit normal, i.e. quantities which vanish when contracted with  $n^\mu$ .

Geometrically they measure how ‘normal’ the coordinate  $r$  is to the hypersurfaces: the choice  $N = 1$ ,  $N^\mu = 0$  makes  $r$  a Gaussian normal coordinate, in which case  $n^\mu$  becomes tangent to *geodesics* normal to the hypersurfaces. A quantity that will be of central importance in our analysis is the extrinsic curvature of the hypersurfaces

$$\hat{K}_{\mu\nu} = \hat{\gamma}_\mu^\rho \nabla_\rho n_\nu = \frac{1}{2} \mathcal{L}_n \hat{\gamma}_{\mu\nu}, \quad (37)$$

where  $\mathcal{L}_n$  denotes the Lie derivative with respect to the unit normal  $n^\mu$ . Thus, the extrinsic curvature measures the radial evolution of the induced metric and hence encapsulates all dynamical information of the geometry of the hypersurfaces. In fact, the Riemann tensor of the  $d+1$  dimensional manifold  $M$  can be expressed entirely in terms of the intrinsic (i.e. Riemannian) and extrinsic curvatures of the hypersurfaces  $\Sigma_r$  via the so called *Gauss–Codacci* equations

$$\begin{aligned} \hat{\gamma}_\mu^\alpha \hat{\gamma}_\nu^\beta \hat{\gamma}_\rho^\gamma \hat{\gamma}_\sigma^\delta R_{\alpha\beta\gamma\delta} &= \hat{R}_{\mu\nu\rho\sigma} + \hat{K}_{\mu\sigma} \hat{K}_{\nu\rho} - \hat{K}_{\mu\rho} \hat{K}_{\nu\sigma}, \\ \hat{\gamma}_\nu^\rho n^\sigma R_{\rho\sigma} &= \hat{\nabla}_\mu \hat{K}_\nu^\mu - \hat{\nabla}_\nu \hat{K}_\mu^\mu. \end{aligned} \quad (38)$$

These purely geometric equations exhibit most explicitly the implications of the bulk  $(d+1)$ -dimensional geometry for the geometry of the hypersurfaces. For, instance, one sees immediately that conformal flatness of the bulk manifold implies very strong constraints on the extrinsic curvature of the radial slices. The case of interest to us here is, of course, the case of an Einstein bulk manifold. A little manipulation of the Gauss–Codacci equations brings them in the following form, most suitable to exhibit the consequences of  $M$  being Einstein; we stress that the following equations are purely geometric:

$$\begin{aligned} \hat{K}^2 - \hat{K}_{\mu\nu} \hat{K}^{\mu\nu} &= \hat{R} + 2G_{\mu\nu} n^\mu n^\nu, \\ \hat{\nabla}_\mu \hat{K}_\nu^\mu - \hat{\nabla}_\nu \hat{K}_\mu^\mu &= G_{\rho\sigma} \hat{\gamma}_\nu^\rho n^\sigma, \\ \mathcal{L}_n \hat{K}_{\mu\nu} + \hat{K} \hat{K}_{\mu\nu} - 2\hat{K}_\mu^\rho \hat{K}_{\rho\nu} &= \hat{R}_{\mu\nu} - \hat{\gamma}_\mu^\rho \hat{\gamma}_\nu^\sigma R_{\rho\sigma}. \end{aligned} \quad (39)$$

These equations become dynamical once we use Einstein’s equations to replace the Einstein tensor with the matter stress tensor. When  $M$  is both conformally flat and Einstein they can be solved exactly [27]. Note that conformal flatness is automatic if  $M$  is three dimensional. However, solving these equations in general for an arbitrary Einstein manifold is equivalent to solving Einstein’s equations and, therefore, far from trivial.

The ADM formalism allows us to express the bulk action in terms of transverse quantities as

$$S = -\frac{1}{2\kappa^2} \int_M d^{d+1}x \sqrt{\hat{\gamma}} N (\hat{R} + \hat{K}^2 - \hat{K}_{\mu\nu} \hat{K}^{\mu\nu} - 2\kappa^2 \mathcal{L}_m). \quad (40)$$

The canonical momenta can now be defined in the standard fashion<sup>7</sup>

$$\pi^{\mu\nu} \equiv \frac{\delta L}{\delta \dot{\gamma}_{\mu\nu}} = -\frac{1}{2\kappa^2} \sqrt{\hat{\gamma}} (\hat{K} \hat{\gamma}^{\mu\nu} - \hat{K}^{\mu\nu}), \quad \pi^I \equiv \frac{\delta L}{\delta \dot{\Phi}_I}, \quad (41)$$

where  $\Phi_I$  is a generic matter field and the Lagrangian  $L$  is defined as usual by  $S = \int dr L$ . In particular, the canonical momenta conjugate to the lapse and shift functions vanish identically, and hence the corresponding equations of motion in the canonical formalism become constraints, which are precisely the first two equations in (39).

Let us finally consider the on-shell gravitational action, as it is precisely this quantity that becomes the generating functional of connected correlation functions of the dual field theory on the boundary. From Einstein's equation and (39) it follows that

$$S_{\text{on-shell}} = -\frac{1}{\kappa^2} \int_{r=r_0}^{r=r_1} dr d^d x \sqrt{\hat{\gamma}} N [\hat{R} + \kappa^2 (n^\mu n^\nu \tilde{T}_{\mu\nu} - \mathcal{L}_m)], \quad (42)$$

where the boundary is located at  $r = r_1$  and  $r_0$  ( $r_0 < r_1$  by our definition of the unit normal) defines a hypersurface in the interior of  $M$ . As mentioned above, there always exists an  $r_0$  sufficiently close to  $r_1$  such that the above expression for the on-shell action is well-defined, but there may not exist an  $r_0$  such that the integration from  $r_0$  to  $r_1$  covers the entire manifold. However, this issue is irrelevant for the near boundary analysis. The on-shell action is a functional of the boundary values of the fields  $\hat{\gamma}(r_1, x)$  and  $\Phi_I(r_1, x)$ . The corresponding momenta on  $\Sigma_{r_1}$  are then obtained from the on-shell action by

$$\pi^{\mu\nu}(r_1, x) = \frac{\delta S_{\text{on-shell}}}{\delta \hat{\gamma}_{\mu\nu}(r_1, x)}, \quad \pi^I(r_1, x) = \frac{\delta S_{\text{on-shell}}}{\delta \Phi_I(r_1, x)}. \quad (43)$$

However, asymptotically AdS spaces are non-compact and the boundary is located at  $r_1 = \infty$ . So the above expressions for the on-shell action and canonical momenta evaluated at  $r_1$  contain divergences due to the infinite volume of the bulk manifold. The advantage of the Hamiltonian formulation is that these expressions hold identically for any hypersurface  $\Sigma_r$  defined by any *finite*  $r$ .

This is then a good point to detail the philosophy of the new approach.

- (1) Using the Hamiltonian formalism we have arrived at a manifestly covariant expression for the canonical momenta evaluated on an arbitrary hypersurface  $\Sigma_r$  for finite  $r$ . In particular, the momenta are *functionals of the bulk fields on  $\Sigma_r$* .
- (2) Using Einstein's equations in the Gauss–Codacci relations (39) – together with any extra equations of motion for matter fields – we obtain a set of second order differential equations for the induced metric and the other bulk fields evaluated on  $\Sigma_r$ .

---

<sup>7</sup>One may consider adding extra *finite* local boundary terms in the action (32). These would result in additional terms in the momenta and finally lead to additional contact terms in (holographically computed) correlators. The addition of such boundary terms is the counterpart of finite local counterterms related to the scheme dependence of the boundary QFT.

- (3) This set of second order ordinary differential equations is then turned into a set of *first order functional partial differential equations* by expressing the radial derivative as a functional derivative. The crucial point here is that the canonical momenta are essentially the  $r$ -derivative of the corresponding bulk fields (up to issues relating to gauge fixing to be discussed below) and we have just seen that the momenta are functionals of the bulk fields on  $\Sigma_r$ . Hence

$$\frac{d}{dr} = \int d^d x 2\hat{K}_{\mu\nu}[\hat{\gamma}, \Phi] \frac{\delta}{\delta \hat{\gamma}_{\mu\nu}} + \int d^d x \dot{\Phi}_I[\hat{\gamma}, \Phi] \frac{\delta}{\delta \Phi_I}. \quad (44)$$

This step is reminiscent, essentially equivalent, to the theorem of Jacobi [7] in the Hamilton–Jacobi theory of classical mechanics, where one expresses the momenta as functional derivatives of the on-shell action as above, but then derives a partial differential equation for the on-shell action. This is precisely the approach followed in [9]. However, we derive functional PDEs for the momenta and this is advantageous as we will discuss momentarily.

- (4) The set of first order functional PDEs thus obtained are, of course, much harder to solve than the original set of second order ODEs<sup>8</sup>, but this representation of the problem is most suitable for the near-boundary analysis in asymptotically AdS spaces, where the bulk fields satisfy prescribed but arbitrary Dirichlet boundary conditions:

$$\hat{\gamma}_{\mu\nu} \sim e^{2r} \hat{\gamma}_{(0)\mu\nu}(x), \quad \Phi_I \sim e^{(\Delta_I - d)r} \phi_{(0)I}(x) \quad (45)$$

as  $r \rightarrow \infty$ , where  $\Delta_I$  is the scaling dimension of the operator dual to the bulk field  $\Phi_I$ . Provided these asymptotics hold<sup>9</sup>, the asymptotic form of the functional representation of the radial derivative is very suggestive:

$$\partial_r \sim \int d^d x 2\hat{\gamma}_{\mu\nu} \frac{\delta}{\delta \hat{\gamma}_{\mu\nu}} + \int d^d x (\Delta_I - d) \Phi_I \frac{\delta}{\delta \Phi_I}. \quad (46)$$

Not surprisingly, this is the total dilatation operator,  $\delta_D$ , of the theory, which appears as a consequence of the well-defined scale transformation rules the fields obey asymptotically. From the point of view of the boundary field theory, this is precisely the Callan–Symanzik equation obeyed by the renormalized one-point functions in the presence of sources, which, as we will see, are related to the canonical momenta. In the spirit of perturbation theory then, it is natural to *expand the momenta in eigenfunctions of the total dilatation operator* and solve the functional PDEs ‘perturbatively’, i.e. asymptotically, while preserving covariance. This is in contrast to the method of holographic renormalization,

<sup>8</sup>In classical mechanics – where the PDEs are not functional – it is often easier to solve the PDEs (either for the action, i.e. the Hamilton–Jacobi equation, or for the momenta) and, as a result solve Hamilton’s equations. This amounts to the ‘inverse method of characteristics’, as Hamilton’s equations are just the characteristic equations for the Hamilton–Jacobi equation.

<sup>9</sup>When  $\Delta = d/2$  the leading asymptotics of the bulk fields are of the form  $r \exp(-dr/2)$ . In those cases the functional representation of the radial derivative must be modified [24], but the above procedure for the asymptotic analysis applies equally well.

where one is seeking asymptotic expansions of the bulk fields using the distance from the AdS boundary as a small parameter – thus explicitly breaking bulk covariance.

- (5) In contrast to previous methods, our focus here is on the canonical momenta and *not* the on-shell action. From the field theory point of view, this is to say we are interested in the exact one-point functions – as opposed to the partition function in the presence of sources. The connection between the canonical momenta and the one-point functions is surprisingly simple in our formalism. On the regulating surface  $\Sigma_r$  with  $r$  finite, the (unrenormalized but regulated) one-point functions are given by the AdS/CFT prescription as the functional derivative of the on-shell action with respect to the corresponding source, i.e. bulk field. Let us show this for the case of matter fields (the derivation for gravity is the same). The variation of the action is given by

$$\delta S = \frac{\partial L}{\partial \dot{\Phi}_I} \delta \Phi_I|' + \int^r dr \delta \Phi_I \left[ \frac{\partial L}{\partial \Phi_I} - \partial_r \left( \frac{\partial L}{\partial \dot{\Phi}_I} \right) \right], \quad (47)$$

where we assumed (without loss of generality) that the matter fields have a standard kinetic term and we gauge fixed as in (50). The second term is just the Euler–Lagrange equation and thus vanishes on-shell. We therefore obtain,

$$\frac{\delta S_{\text{on-shell}}}{\delta \Phi_I} = \pi^I(r), \quad (48)$$

where we used (41). The left-hand side is, by the AdS/CFT dictionary, the regulated 1-point function in the presence of sources. We now argue that a similar connection holds for the *renormalized* one-point functions as well. The renormalized one-point functions are defined to be the one-point functions one obtains from the renormalized action. This is in turn the on-shell action plus a set of covariant counterterms which remove the divergences of the on-shell action as  $r \rightarrow \infty$ . Suppose now these covariant counterterms for the on-shell action are constructed. Taking the functional derivative with respect to the appropriate bulk field they lead to covariant terms which when added to the regularized momenta must – by construction – remove all the potential singularities from the canonical momenta. We thus need to identify the singular part of canonical momentum and remove it. The main constraint is that the subtraction should be covariant. This is done by expanding the momentum in terms of (covariant) eigenfunctions of the dilatation operator and observing that the divergent terms have eigenvalues less than the dimension  $\Delta$  of the dual operator. To summarize, we have the very general result

$$\boxed{\langle \hat{T}_{\mu\nu} \rangle_{\text{ren}} = -\frac{1}{\kappa^2} (\hat{K}_{(d)\mu\nu} - \hat{K}_{(d)} \hat{\gamma}_{\mu\nu}), \quad \langle \mathcal{O}_I \rangle_{\text{ren}} = \frac{1}{\sqrt{\hat{\gamma}}} \pi^I_{(\Delta_I)},} \quad (49)$$

The terms on the right hand sides have the (engineering) dimension indicated by their subscript. These would have also been their dilatation eigenvalues in the



absence of conformal anomalies and RG running. In most examples, the bulk theory involves non-trivially only fields of the gauged supergravity obtained by reducing the 10d supergravity over a compact manifold. In such cases, the bulk fields correspond to operators of protected dimensions and thus the coefficients in (49) fail to be eigenfunctions of the dilatation operator only because of the conformal anomaly. As we shall see explicitly in the examples, the conformal anomaly induces an inhomogeneous term in the dilatation transformation of the coefficients in (49). We further note that these coefficients are not completely determined by the asymptotic analysis; they are the counterparts of the undetermined coefficients of the near-boundary analysis. It is therefore redundant to first construct covariant counterterms for the on-shell action and then use them to obtain the renormalized one-point functions, since the equations of motion can be solved for the momenta directly, and these are all one needs to obtain the one-point functions.

- (6) Although, as we just argued, it is not necessary to compute covariant counterterms for the on-shell action in order to obtain renormalized correlation functions, one *can* construct them as a secondary step in our method, and in fact more efficiently than previous methods. This is done by constructing a differential equation – essentially equivalent to the Hamilton–Jacobi equation – for the on-shell action which we then solve in parallel to the equations for the momenta. Explicit examples will be presented below.

In previous works where a Hamiltonian approach was used [9], [22], [14] a central point of the analysis was the solution of the Hamilton–Jacobi (HJ) equation for the on-shell action. In this context, the HJ equation is a functional PDE for the on-shell action which can be solved by inserting an ansatz for the on-shell action in it. By requiring that terms with different number of derivatives cancel separately one gets a number of equations, the descent equations, that can be solved to determine the unknown functions in the ansatz. In the presence of scalars, the equations were further organized in [22] according to the number of scalar fields they contain. The resulting equations are not in general equivalent to the ones in the standard approach. This is due to the fact that the scalar fields are treated differently than in the standard holographic renormalization. Recall that in the standard approach the equations are solved by using the distance from the boundary as a small parameter with all sources being unconstrained. The expansion in the number of scalar fields requires that all scalar fields are (equally) small and for this to be the case the Dirichlet data (QFT sources) should be tuned to be (appropriately) small. This is rather unnatural since on the QFT side all sources are unconstrained and of order one and in general can lead to erroneous results. In simple examples, such as the ones studied in the literature, there is no obstruction in considering the sources small and the results so obtained are in agreement with results obtained via the standard method. An alternative approach that overcomes these issues is to organize the terms in the HJ equation

according to their dilatation weight. This yields equations that are equivalent to the ones in the standard holographic renormalization method. Solving the HJ equation for the on-shell action leads to some of the same simplifications we find here. For instance, the covariant counterterm action is derived easier. On the other hand, the use of an ansatz for the on-shell action (instead of constructively obtaining the most general solution) as well as various sign ambiguities make the method less rigorous than the standard approach. More importantly, focusing on the canonical momenta instead of the on-shell action appears to be the most economic way to proceed.

- (7) Apart from an elegant framework for the general asymptotic analysis, this formalism provides a most efficient way to calculate *correlation functions* of the boundary field theory holographically. As we have just seen this amounts to determining the renormalized canonical momenta as functionals of arbitrary bulk fields, i.e. as functionals of arbitrary sources. To determine 2-point functions we only need to determine the momenta in terms of the source at linearized level. Furthermore, the contribution of the counterterms to 2-point functions can also be determined directly from the *linearized* analysis, following the discussion in the previous point. A similar discussion applies also to  $n$ -point functions ( $n > 2$ ). This leads to a significant simplification of the computation of correlation functions. Details will appear elsewhere [24].

**Gauge fixing.** Before we carry out the near boundary analysis for pure AdS gravity and gravity coupled to scalars following the above prescription, let us fix the gauge freedom associated with the shift and lapse functions by setting  $N^\mu = 0$  and  $N = 1$ . The bulk metric then takes the form<sup>10</sup>

$$ds^2 = dr^2 + \gamma_{ij}(r, x)dx^i dx^j, \quad (50)$$

where  $i, j = 1, \dots, d$  are indices along the hypersurfaces. The extrinsic curvature becomes

$$K_{ij} = \frac{1}{2} \dot{\gamma}_{ij} \quad (51)$$

where the dot denotes differentiation with respect to  $r$ . The non-vanishing components of the Christoffel symbol are

$$\Gamma_{ij}^{d+1} = -K_{ij}, \quad \Gamma_{d+1j}^i = K_j^i, \quad G_{jk}^i. \quad (52)$$

The gravitational field equations (39) take the form

$$\begin{aligned} K^2 - K_{ij}K^{ij} &= R + 2\kappa^2 \tilde{T}_{d+1d+1}, \\ \nabla_i K_j^i - \nabla_j K &= \kappa^2 \tilde{T}_{jd+1}, \\ \dot{K}_j^i + K K_j^i &= R_j^i - \kappa^2 \left( \tilde{T}_j^i + \frac{1}{1-d} \tilde{T}_\sigma^\sigma \delta_j^i \right). \end{aligned} \quad (53)$$

---

<sup>10</sup>All tensors are transverse and so we drop the hats from now on.

$\dot{K}_j^i$  here stands for  $\frac{d}{dr}(\gamma^{ik}K_{kj})$ . An additional equation for the on-shell action can be derived as promised. Since

$$\dot{S}_{\text{on-shell}} = L = -\frac{1}{\kappa^2} \int_{\Sigma_r} d^d x \sqrt{\gamma} [R + \kappa^2 (\tilde{T}_{d+1d+1} - \mathcal{L}_m)], \quad (54)$$

we can obtain an expression for  $S_{\text{on-shell}}$  if we write the integrand as the derivative of some (covariant) quantity. This is achieved by introducing a covariant variable  $\lambda$  and writing

$$S_{\text{on-shell}} = -\frac{1}{\kappa^2} \int_{\Sigma_r} d^d x \sqrt{\gamma} (K - \lambda). \quad (55)$$

Taking the trace of the third equation in (53) we determine that  $\lambda$  satisfies

$$\dot{\lambda} + K\lambda - \kappa^2 \left( \mathcal{L}_m - \frac{1}{1-d} \tilde{T}_\sigma^\sigma \right) = 0. \quad (56)$$

#### 4.1 Pure gravity case

We will now demonstrate the method of Hamiltonian holographic renormalization for pure gravity with a negative cosmological constant<sup>11</sup>  $\Lambda = d(1-d)/2$ . The equations of motion reduce to

$$\begin{aligned} K^2 - K_{ij}K^{ij} &= R + d(d-1), \\ \nabla_i K_j^i - \nabla_j K &= 0, \\ \dot{K}_j^i + K K_j^i &= R_j^i + d\delta_j^i. \end{aligned} \quad (57)$$

The on-shell action is determined from the equation

$$\dot{\lambda} + K\lambda = d. \quad (58)$$

We will expand the extrinsic curvature and  $\lambda$  in eigenfunctions of the dilatation operator, which now takes the form

$$\delta_D = \int d^d x 2\gamma_{ij} \frac{\delta}{\delta\gamma_{ij}}. \quad (59)$$

Then

$$\begin{aligned} K_j^i[\gamma] &= K_{(0)}^i{}_j + K_{(2)}^i{}_j + \cdots + K_{(d)}^i{}_j + \tilde{K}_{(d)}^i{}_j \log e^{-2r} + \cdots, \\ \lambda[\gamma] &= \lambda_{(0)} + \lambda_{(2)} + \cdots + \lambda_{(d)} + \tilde{\lambda}_{(d)} \log e^{-2r} + \cdots, \end{aligned} \quad (60)$$

where

$$\begin{aligned} \delta_D K_{(n)}^i{}_j &= -n K_{(n)}^i{}_j, \quad n < d, \\ \delta_D \tilde{K}_{(d)}^i{}_j &= -d \tilde{K}_{(d)}^i{}_j, \end{aligned} \quad (61)$$

---

<sup>11</sup> $S \sim \int (R - 2\Lambda)$ .

$$\begin{aligned}
\delta_D K_{(d)j}^i &= -d K_{(d)j}^i - 2 \tilde{K}_{(d)j}^i \\
\delta_D \lambda_{(n)} &= -n \lambda_{(n)}, \quad n < d, \\
\delta_D \tilde{\lambda}_{(d)} &= -d \tilde{\lambda}_{(d)}, \\
\delta_D \lambda_{(d)} &= -d \lambda_{(d)} - 2 \tilde{\lambda}_{(d)}.
\end{aligned}$$

The inhomogeneous transformations of  $K_{(d)j}^i$  and  $\lambda_{(d)}$ , which follow immediately from the relation between the radial derivative and the dilatation operator, are due to the conformal anomaly.

Before we proceed to determine these coefficients from the equations of motion, let us exhibit the equivalence of this covariant expansion in eigenfunctions of the dilatation operator to the asymptotic expansion of the induced metric in the standard holographic renormalization method. There the induced metric is expanded in  $\rho = \exp(-2r)$  as

$$\gamma_{ij} = \frac{1}{\rho} [g_{(0)ij} + \rho g_{(2)ij} + \cdots + \rho^{d/2} g_{(d)ij} + \rho^{d/2} \log \rho h_{(d)ij} + \cdots] \quad (62)$$

Hence

$$\begin{aligned}
\frac{1}{2} \dot{\gamma}_{ij} &= \frac{1}{\rho} g_{(0)ij} - \rho g_{(4)ij} + \cdots \\
&+ \rho^{(d/2-1)} \left[ \left( 1 - \frac{d}{2} \right) g_{(d)ij} - h_{(d)ij} \right] \\
&+ \rho^{(d/2-1)} \log \rho \left( 1 - \frac{d}{2} \right) h_{(d)ij} + \cdots .
\end{aligned} \quad (63)$$

However, each term in the covariant expansion of the extrinsic curvature is a functional of the induced metric  $\gamma_{ij}$ . Using the expansion (62) of the metric we can functionally expand the eigenfunctions of the dilatation operator as

$$\begin{aligned}
K_{(0)ij}[\gamma] &= \gamma_{ij} \\
&= \frac{1}{\rho} [g_{(0)ij} + \rho g_{(2)ij} + \cdots + \rho^{d/2} g_{(d)ij} + \rho^{d/2} \log \rho h_{(d)ij} + \cdots], \\
K_{(2)ij}[\gamma] &= K_{(2)ij}[g_{(0)}] + \rho \int d^d x g_{(2)kl} \frac{\delta K_{(2)ij}}{\delta g_{(0)kl}} + \cdots, \\
&\vdots \\
K_{(d)ij}[\gamma] &= \rho^{(d/2-1)} K_{(d)ij}[g_{(0)}] + \cdots, \\
\tilde{K}_{(d)ij}[\gamma] &= \rho^{(d/2-1)} \tilde{K}_{(d)ij}[g_{(0)}] + \cdots.
\end{aligned} \quad (64)$$

Inserting these expressions in the covariant expansion for  $K_{ij}$  and comparing with (63) we determine

$$\begin{aligned}
K_{(0)ij}[g_{(0)}] &= g_{(0)ij}, \\
K_{(2)ij}[g_{(0)}] &= -g_{(2)ij}[g_{(0)}], \\
&\vdots \\
K_{(n)ij}[g_{(0)}] &= -\frac{n}{2}g_{(n)ij}[g_{(0)}] + \text{lower}, \\
&\vdots \\
K_{(d)ij}[g_{(0)}] &= -\frac{d}{2}g_{(d)ij}[g_{(0)}] - h_{(d)ij}[g_{(0)}] + \text{lower}, \\
\tilde{K}_{(d)ij}[g_{(0)}] &= -\frac{d}{2}h_{(d)ij}[g_{(0)}],
\end{aligned} \tag{65}$$

where ‘lower’ stands for terms involving functional derivatives with respect to  $g_{(0)ij}$  of lower order coefficients  $g_{(k)ij}[g_{(0)}]$ . For  $d=4$ , for example,

$$K_{(4)ij}[g_{(0)}] = -2g_{(4)ij}[g_{(0)}] - h_{(4)ij}[g_{(0)}] + \int d^4x g_{(2)kl} \frac{\delta g_{(2)ij}[g_{(0)}]}{\delta g_{(0)kl}}. \tag{66}$$

Thus there is a one-to-one correspondence between the terms in the asymptotic expansion of holographic renormalization and our covariant expansion in eigenfunctions of the dilatation operator. In particular, the non-local terms in the two expansions are related, whereas the coefficients of the logarithms which are related to the conformal anomaly are just proportional to each other. This completes our demonstration of the equivalence of the two methods.

The new formulation, however, is advantageous over the standard method in that the on-shell action is expressed entirely in terms of the extrinsic curvature coefficients, for arbitrary  $d$ . Furthermore, the one-point function in the presence of sources is also expressed simply in terms of one of the extrinsic curvature coefficients. The asymptotic analysis is done once, for all  $d$ , resulting in generic recursion relations for the extrinsic curvature coefficients.

To complete the near boundary analysis one then just needs to solve the recursion relations for a given dimension  $d$ . The key ingredient in our method which allows for these improvements is the functional relation between the canonical momenta and the on-shell action, namely

$$\pi^{ij} = -\frac{1}{2\kappa^2}\sqrt{\gamma}(K\gamma^{ij} - K^{ij}) = \frac{\delta S_{\text{on-shell}}}{\delta \gamma_{ij}} \tag{67}$$

or

$$K\gamma^{ij} - K^{ij} = \frac{2}{\sqrt{\gamma}} \frac{\delta}{\delta\gamma_{ij}} \int_{\Sigma_r} d^d x \sqrt{\gamma} (K - \lambda). \quad (68)$$

Inserting the covariant expansions for  $K_j^i$  and  $\lambda$  we can relate the coefficients of the on-shell action to those of the extrinsic curvature as

$$\begin{aligned} K_{(2n)j}^i &= \lambda_{(2n)} \delta_j^i - \frac{2}{\sqrt{\gamma}} \int d^d x \sqrt{\gamma} \gamma_{kj} \frac{\delta}{\delta\gamma_{ik}} (K_{(2n)} - \lambda_{(2n)}), \quad 0 \leq n \leq \frac{d}{2}, \\ \tilde{K}_{(d)j}^i &= \tilde{\lambda}_{(d)} \delta_j^i - \frac{2}{\sqrt{\gamma}} \int d^d x \sqrt{\gamma} \gamma_{kj} \frac{\delta}{\delta\gamma_{ik}} (\tilde{K}_{(d)} - \tilde{\lambda}_{(d)}). \end{aligned} \quad (69)$$

The trace of these equations then gives

$$(1 + \delta_D) K_{(2n)} = (d + \delta_D) \lambda_{(2n)}, \quad 0 \leq n \leq \frac{d}{2}, \quad (1 + \delta_D) \tilde{K}_{(d)} = (d + \delta_D) \tilde{\lambda}_{(d)}. \quad (70)$$

Since we know how the coefficients transform under the dilatation operator, these relations completely determine  $\lambda$  in terms of the trace of the extrinsic curvature. Namely we obtain the significant result

$$\lambda_{(2n)} = \frac{(2n-1)}{(2n-d)} K_{(2n)}, \quad 0 \leq n \leq \frac{d}{2} - 1, \quad \tilde{\lambda}_{(d)} = \frac{d-1}{2} K_{(d)}, \quad \tilde{K}_{(d)} = 0. \quad (71)$$

The coefficient  $K_{(2n)j}^i$  are only determined for  $n < d/2$ . If one does the computation for general  $d$  then the corresponding expression has a first order pole at  $d = 2n$ . A short computation using (69) shows that the residue of the pole is exactly  $\tilde{K}_{(d)j}^i$ , i.e. the coefficient of the logarithmic term in  $d$  dimensions

$$\tilde{K}_{(d)j}^i = \lim_{n \rightarrow d/2} \left( \left( n - \frac{d}{2} \right) K_{(2n)j}^i \right). \quad (72)$$

In practice one can also use this result in order to compute  $K_{(d-2)j}^i$  in  $d$  dimensions from  $\tilde{K}_{(d-2)j}^i$  in  $d-2$  dimensions.

We thus arrive at a general closed form expression for the covariant counterterm action that renders the on-shell action finite:

$$S_{ct} = \frac{(1-d)}{\kappa^2} \int_{\rho=\epsilon} d^d x \sqrt{\gamma} \left[ \sum_{m=0}^{\frac{d}{2}-1} \frac{1}{(2m-d)} K_{(2m)} + \frac{1}{2} K_{(d)} \log \epsilon \right] \quad (73)$$

The rest of the analysis is now straightforward. First, by direct substitution of the covariant expansion of the extrinsic curvature into the first equation in (57) one finds

a recursive relation for the traces of the extrinsic curvature coefficients, namely

$$\boxed{\begin{aligned} K_{(2)} &= \frac{R}{2(d-1)}, \\ K_{(2n)} &= \frac{1}{2(d-1)} \sum_{m=1}^{n-1} [K_{(2m)ij} K_{(2n-2m)}^{ij} - K_{(2m)} K_{(2n-2m)}], \quad 2 \leq n \leq \frac{d}{2} \end{aligned}} \quad (74)$$

Finally, inserting the values of  $\lambda_{(2n)}$  and the traces of the extrinsic curvature we have determined in (71) and (74) into the functional relation (69) one can evaluate all coefficients recursively. In doing so, one sees that considerable simplifications occur upon using the second equation in (57), which implies

$$\nabla_i K_{(2n)j}^i - \nabla_j K_{(2n)} = 0, \quad 0 \leq n \leq \frac{d}{2}, \quad \nabla_i \tilde{K}_{(d)j}^i - \nabla_j \tilde{K}_{(d)} = 0. \quad (75)$$

Note that although  $K_{(d)j}^i$  is non-local in general, its trace is local as follows from (74). Carrying out the above procedure is straightforward but the result becomes of forbidding complexity as one goes up in dimension. The algorithm, however, could be implemented in a computer code which would in principle calculate the counterterms and the holographic Weyl anomaly for any dimension. For illustrative purposes we quote the results for up to four dimensions ( $d = 2$  and  $d = 4$ ):.

$$\begin{aligned} K_j^i[\gamma] &= \delta_j^i + K_{(2)j}^i + \dots, \\ K[\gamma] &= d + P + \dots, \\ S_{\text{ct}} &= \frac{(d-1)}{\kappa^2} \int_{\rho=\epsilon} d^2x \sqrt{\gamma} \left[ 1 - \frac{1}{4} R \log \epsilon \right]. \\ K_j^i[\gamma] &= \delta_j^i + P_j^i + \frac{1}{2} \left[ \frac{1}{2} (P^{kl} P_{kl} - P^2) \delta_j^i \right. \\ &\quad \left. - \frac{1}{(d-2)} (2R^i_{kjl} P^{kl} - P R_j^i + \square P_j^i - \nabla^i \nabla_j P) \right] \log \epsilon + K_{(4)j}^i + \dots \\ K[\gamma] &= d + P + \frac{1}{2(d-1)} (P^{kl} P_{kl} - P^2) + \dots \\ S_{\text{ct}} &= \frac{(d-1)}{\kappa^2} \int_{\rho=\epsilon} d^4x \sqrt{\gamma} \left[ 1 + \frac{1}{(d-2)} P - \frac{1}{4(d-1)} (P^{kl} P_{kl} - P^2) \log \epsilon \right] \end{aligned}$$

where we have introduced the sectional curvature tensor

$$P_{ij} = \frac{1}{(d-2)} \left( R_{ij} - \frac{1}{2(d-1)} R \gamma_{ij} \right) \quad (76)$$

which transforms under Weyl rescalings of the metric  $\delta\gamma_{ij} = -2\gamma_{ij}\delta\sigma$  as  $\delta P_{ij} = \nabla_i \nabla_j \delta\sigma$ .

## 4.2 Gravity coupled to scalars

Having carried out in detail the near boundary analysis for pure AdS gravity in our formalism, we will now briefly describe how the analysis can be generalized to include scalars. In this case the matter action takes the form

$$S_m = \int_M d^{d+1}x \sqrt{g} \left[ \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi_I \partial_\nu \Phi_I + V(\Phi_I) \right]. \quad (77)$$

Along with the gravitational field equations (53) and equation (56) for the on-shell action, we now have the equations of motion for the scalar fields

$$\ddot{\Phi}_I + K \dot{\Phi}_I + \square \Phi_I - \frac{\partial}{\partial \Phi_I} V(\Phi) = 0. \quad (78)$$

In terms of the canonical momenta<sup>12</sup>  $\pi^I = \dot{\Phi}_I$ ,

$$\dot{\pi}^I + K \pi^I + \square \Phi_I - \frac{\partial}{\partial \Phi_I} V(\Phi) = 0. \quad (79)$$

Next we expand the canonical momenta and on-shell action in eigenfunctions of the dilatation operator

$$\delta_D = \int d^d x 2\gamma_{ij} \frac{\delta}{\delta \gamma_{ij}} + \int d^d x (\Delta_I - d) \Phi_I \frac{\delta}{\delta \Phi_I}. \quad (80)$$

In addition to the expansions (60) we now have expansions for the canonical momenta of the scalar fields

$$\pi^I[\gamma, \Phi] = \sum_{d-\Delta_I \leq s < \Delta_I} \pi_{(s)}^I + \pi_{(\Delta_I)}^I + \tilde{\pi}_{(\Delta_I)}^I \log e^{-2r} + \dots \quad (81)$$

The crucial difference is that the momenta now depend on *all* bulk fields and not just the induced metric. Moreover, depending on the scaling dimension  $\Delta_I$ , the eigenvalues of the dilatation operator may not be integers anymore. The analysis is exactly analogous to that for pure AdS gravity, making essential use of the functional relations (67) and

$$\pi^I = -\frac{1}{\kappa^2} \frac{1}{\sqrt{\gamma}} \frac{\delta}{\delta \Phi_I} \int d^d x \sqrt{\gamma} (K - \lambda), \quad (82)$$

which imply the key relation

$$(1 + \delta_D) K + \kappa^2 (\Delta_I - d) \pi^I \Phi_I = (d + \delta_D) \lambda. \quad (83)$$

As for pure AdS gravity, this can be used to express the coefficients of  $\lambda$  in terms of those of the momenta. Then, inserting  $\lambda$  into (67) and (82), the canonical momenta are determined iteratively.

---

<sup>12</sup>Strictly speaking, the momenta are densities and should include a factor of  $\sqrt{\gamma}$  as we defined them earlier, see for instance [30]. Nevertheless, we will drop (with due care) this factor in this section as this results in simpler equations.



As an illustration, consider the case of two scalar fields,  $\Phi$  and  $\Sigma$ , both of scaling dimension  $\Delta = 3$  in  $d = 4$  and with potential that has a critical point at  $\Phi = \Sigma = 0$ . The most general potential compatible with these requirements is

$$V(\Phi, \Sigma) = \sum_{n=0}^{\infty} \sum_{m=0}^n \kappa^{n-2} V_{(m,n-m)} \Phi^m \Sigma^{n-m}, \quad (84)$$

where  $V_{(0,0)} = \Lambda/\kappa^2$  is the cosmological constant,  $V_{(0,1)} = V_{(1,0)} = 0$ , i.e. there are no linear couplings,  $V_{(1,1)} = 0$  and  $V_{(2,0)} = V_{(0,2)} = -3$ , i.e. the quadratic terms are diagonal in  $\Phi$  and  $\Sigma$  and both have mass  $m^2 = \Delta(\Delta - d) = -3$ , and all other coupling  $V_{(m,m-n)}$  are arbitrary. The iterative approach determines the following on-shell action:

$$\begin{aligned} S_{\text{on-shell}} = & -\frac{1}{\kappa^2} \int_{\rho=\epsilon} d^4x \sqrt{\gamma} \left\{ \frac{d-1}{d-2} P - \frac{1}{4} [P_{ij} P^{ij} - P^2 \right. \\ & - \kappa^2 \Phi(\square + P)\Phi - \kappa^2 \Sigma(\square + P)\Sigma] \log \epsilon + K_{(4)} \\ & \left. - \lambda_{(4)} + \dots \right\} - \int_{\rho=\epsilon} d^4x \sqrt{\gamma} W(\Phi, \Sigma) \end{aligned} \quad (85)$$

where the “superpotential” is given by

$$\begin{aligned} W(\Phi, \Sigma) &= \frac{1}{\kappa^2} (d-1) + \frac{1}{2} (\Phi^2 + \Sigma^2) \\ &+ \frac{\kappa}{d-3} (V_{(3,0)} \Phi^3 + V_{(2,1)} \Phi^2 \Sigma + V_{(1,2)} \Phi \Sigma^2 + V_{(0,3)} \Sigma^3) \\ &+ \kappa^2 \left[ \left( \frac{1}{2} V_{(4,0)} - \frac{1}{4(d-3)^2} (9V_{(3,0)}^2 + V_{(1,2)}^2) + \frac{d}{16(d-1)} \right) \Phi^4 \right. \\ &+ \left( \frac{1}{2} V_{(3,1)} - \frac{1}{(d-3)^2} (3V_{(3,0)} V_{(2,1)} + V_{(1,2)} V_{(0,3)}) \right) \Phi^3 \Sigma \\ &+ \left( \frac{1}{2} V_{(2,2)} - \frac{1}{2(d-3)^2} (3V_{(3,0)} V_{(1,2)} \right. \\ &+ 3V_{(0,3)} V_{(2,1)} + 2V_{(2,1)}^2 + 2V_{(1,2)}^2) + \frac{d}{8(d-1)} \left. \right) \Phi^2 \Sigma^2 \\ &+ \left( \frac{1}{2} V_{(0,4)} - \frac{1}{4(d-3)^2} (9V_{(0,3)}^2 + V_{(2,1)}^2) + \frac{d}{16(d-1)} \right) \Sigma^4 \\ &+ \left. \left( \frac{1}{2} V_{(1,3)} - \frac{1}{(d-3)^2} (3V_{(0,3)} V_{(1,2)} + V_{(2,1)} V_{(3,0)}) \right) \Phi \Sigma^3 \right] \log \epsilon \\ &+ W_{(4)} + \dots \end{aligned} \quad (86)$$

A direct computation shows that  $W$  satisfies

$$V(\Phi_I) = \frac{1}{2} \left[ \left( \frac{\partial W}{\partial \Phi_I} \right)^2 - \frac{d\kappa^2}{d-1} W^2 \right]. \quad (87)$$

and thus can be considered as a “superpotential”. The AdS critical point of  $V$  is also a critical point of  $W$ . This together with (87) guarantee gravitational stability of the AdS critical point and of BPS domain-wall solutions of the gauged supergravity [29], [28]. The expression (87) can be derived in general in this formalism, i.e. for arbitrary dimension and scalar fields [9]. In this context one may also view (87) as a differential equation that should be solved to determine the “superpotential”. Since (87) is quadratic in  $W$  it can only determine  $W$  up to a sign. In contrast, our method *guarantees* that the “superpotential” satisfies (87), but nevertheless there is no sign ambiguity in the determination of (86).

So far we have presented a method to solve the first order functional differential equations asymptotically. It would be, of course, desirable to solve these non-linear equations completely for arbitrary sources, but this is beyond present capabilities. However, we can get an idea of the difficulty of the problem by considering the easiest part, namely the superpotential for one scalar field. The full superpotential can be determined (up to sign that is fixed using the asymptotic solution for  $W$ ) by equation (87) which is a first order ODE, as opposed to the first order functional differential equations that determine the rest of the on-shell action. To solve for the superpotential we observe that it is possible to bring (87) into the form of Abel’s equation [19]:

$$y'(\psi) = \frac{v'}{v} y^3 - y^2 - \frac{v'}{v} y + 1 \quad (88)$$

where  $\psi = \sqrt{\frac{d\kappa^2}{d-1}} \phi$ ,  $y = \coth(u)$ ,  $W = v \cosh(u)$ , and  $v$  is related to the potential by  $\frac{2(d-1)}{d\kappa^2} V = -v^2$ . The general solution to this equation is not known, but it can be solved in special cases. For example, the potential

$$V(\psi) = -\frac{d(d-1)}{2\kappa^2} \cosh\left(\frac{2}{3}\psi\right) \quad (89)$$

leads to a soluble equation with solution

$$\begin{aligned} W(\psi) &= \frac{(d-1) \cosh^{1/2}\left(\frac{2}{3}\psi\right) \left( \cosh\left(\frac{2}{3}\psi\right) + \cosh^{1/2}(\gamma) \operatorname{sech}^{1/2}\left(\frac{4}{3}\psi + \gamma\right) \right)}{\kappa^2 \sqrt{1 + \cosh(\gamma) \operatorname{sech}\left(\frac{4}{3}\psi + \gamma\right) + 2 \cosh\left(\frac{2}{3}\psi\right) \cosh^{1/2}(\gamma) \operatorname{sech}^{1/2}\left(\frac{4}{3}\psi + \gamma\right)}}. \end{aligned} \quad (90)$$

Here  $\gamma$  is an arbitrary parameter, and the scaling dimension,  $\Delta$ , of the operator dual to the scalar field is  $2d/3$ . Both the potential and the superpotential have  $\phi = 0$  as a

critical point.  $W$  has an expansion around zero

$$W(\psi) = \frac{d-1}{\kappa^2} \left( 1 + \frac{1}{6}\psi^2 + \frac{1}{27}\tanh(\gamma)\psi^3 + \dots \right). \quad (91)$$

Since  $\Delta = 2d/3$  the term cubic in  $\phi$  has a dilatation eigenvalue  $3(\Delta - d) = -d$  which is exactly the correct order where the asymptotic expansion of the on-shell action breaks down and an undetermined term arises. In this case it happens that there is no logarithmic term, but a new parameter  $\gamma$  appears at the correct order.

## 5 Conclusions

We reviewed in this paper how QFT data is encoded in geometry via the AdS/CFT correspondence and we developed a new, more powerful, calculational method. A central element in the extraction of the QFT data is the form of asymptotic solutions for Einstein and matter field equations. The asymptotic solutions contain, after a number of terms that are uniquely determined in terms of the Dirichlet data, a term that is only partially determined by the asymptotic analysis. At the same order a logarithmic term may appear. We can now summarize the way the QFT data are encoded as follows:

- The undetermined coefficient encodes all correlation functions. To uncover them one needs exact solutions with arbitrary Dirichlet data.
- The relations that the undetermined coefficients satisfy are related to QFT Ward identities, including anomalies.
- The logarithmic term is also related to the conformal anomaly.

The conformal anomaly appears also as a coefficient of the logarithmic divergence of the on-shell action [18]. The relation between logarithmic divergences in the on-shell action and the logarithmic term in the asymptotic expansion of a (free) scalar field has also recently appeared in the math literature [16], [13] in studies related to the so-called Q-curvature<sup>13</sup>. It would be interesting to understand the significance of the Q-curvature in terms of the dual quantum field theory. Let us also emphasize again that in terms of physics it is most important to obtain information about the undetermined coefficients by studying, for instance, the constraints implied by regularity in the interior.

We have presented a new method for obtaining renormalized correlation functions, conformal anomalies and Ward identities of the boundary QFT from geometrical data. The method is a Hamiltonian version of the standard approach but with the radial coordinate playing the role of time. In this approach the renormalized one-point functions of the boundary QFT in the presence of sources are related to the canonical

---

<sup>13</sup>The Q-curvature is a generalization of the scalar curvature in two dimensions: it satisfies analogous conformal transformation properties.

momenta of the bulk fields. The near-boundary expansions of the standard method are replaced by covariant expansions in eigenfunctions of the dilatation operator. This leads to simple closed form expressions for the counterterms and for the renormalized one-point functions in terms of the coefficients of the covariant expansion of the momenta, valid in all dimensions. For the coefficients we derived general recursion relations (also valid in any dimension). This leads to a more efficient algorithm for determining counterterms and one-point functions than in previous works. It would be important to integrate the recursion relation in general. It seems likely that this would require a more intuitive and geometric understanding of the conformal anomaly.

**Acknowledgments.** We would like to thank W. Mück for comments on the first version of this work. The second author is supported by NWO.

## References

- [1] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, Large  $N$  field theories, string theory and gravity, *Phys. Rep.* 323 (2000), 183–386 [arXiv:hep-th/9905111].
- [2] M. T. Anderson, Geometric aspects of the AdS/CFT correspondence, in: *AdS-CFT Correspondence: Einstein Metrics and their Conformal Boundaries*, ed. by Olivier Biquard, IRMA Lect. Math. Theor. Phys. 8, European Math. Soc. Publishing House, Zürich 2005, 1–31 [arXiv:hep-th/0403087].
- [3] V. Balasubramanian and P. Kraus, A stress tensor for anti-de Sitter gravity, *Comm. Math. Phys.* 208 (1999), 413–428 [arXiv:hep-th/9902121].
- [4] M. Bianchi and A. Marchetti, Holographic three-point functions: One step beyond the tradition, arXiv:hep-th/0302019; M. Bianchi, M. Prisco and W. Muck, New results on holographic three-point functions, *J. High Energy Phys.* 11 (2003), 052 [arXiv:hep-th/0310129]; W. Muck and M. Prisco, Glueball scattering amplitudes from holography, arXiv:hep-th/0402068.
- [5] M. Bianchi, D. Z. Freedman and K. Skenderis, How to go with an RG flow, *J. High Energy Phys.* 08 (2001), 041 [arXiv:hep-th/0105276]; Holographic renormalization, *Nucl. Phys. B* 631 (2002), 159–194 [arXiv:hep-th/0112119].
- [6] L. Bonora, P. Pasti and M. Bregola, Weyl Cocycles, *Classical Quantum Gravity* 3 (1986), 635–649.
- [7] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Vol. II: Partial differential equations, Interscience Publishers, New York 1962.
- [8] E. D’Hoker and D. Z. Freedman, Supersymmetric gauge theories and the AdS/CFT correspondence [arXiv:hep-th/0201253].
- [9] J. de Boer, E. Verlinde and H. Verlinde, On the holographic renormalization group, *J. High Energy Phys.* 08 (2000), 003 [arXiv:hep-th/9912012]; J. de Boer, The holographic renormalization group, *Fortschr. Phys.* 49 (2001), 339–358 [arXiv:hep-th/0101026].

- [10] S. de Haro, S. N. Solodukhin and K. Skenderis, Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence, *Comm. Math. Phys.* 217 (2001), 595–622 [arXiv:hep-th/0002230].
- [11] S. Deser and A. Schwimmer, Geometric classification of conformal anomalies in arbitrary dimensions, *Phys. Lett. B* 309 (1993), 279–284 [arXiv:hep-th/9302047].
- [12] C. Fefferman and C. R. Graham, Conformal invariants, in: *Élie Cartan et les mathématiques d’aujourd’hui* (Lyon 1984), *Astérisque* (1985), Numéro hors-série, 95–116.
- [13] C. Fefferman and C. R. Graham, Q-curvature and Poincare metrics, math.DG/0110271.
- [14] M. Fukuma, S. Matsuura and T. Sakai, A note on the Weyl anomaly in the holographic renormalization group, *Progr. Theor. Phys.* 104 (2000), 1089–1108 [arXiv:hep-th/0007062]; J. Kalkkinen and D. Martelli, Holographic renormalization group with fermions and form fields, *Nucl. Phys. B* 596 (2001), 415–438 [arXiv:hep-th/0007234]; J. Kalkkinen, D. Martelli and W. Muck, Holographic renormalisation and anomalies, *J. High Energy Phys.* 04 (2001), 036 [arXiv:hep-th/0103111]; M. Fukuma, S. Matsuura and T. Sakai, Holographic renormalization group, *Progr. Theor. Phys.* 109 (2003), 489–562 [arXiv:hep-th/0212314].
- [15] C. R. Graham, Volume and Area Renormalizations for Conformally Compact Einstein Metrics, math.DG/9909042.
- [16] C. R. Graham and M. Zworski, Scattering matrix in conformal geometry, math.DG/0109089.
- [17] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, Gauge theory correlators from non-critical string theory, *Phys. Lett. B* 428 (1998), 105–114 [arXiv:hep-th/9802109].
- [18] M. Henningson and K. Skenderis, The holographic Weyl anomaly, *J. High Energy Phys.* 07 (1998), 023 [arXiv:hep-th/9806087]; M. Henningson and K. Skenderis, Holography and the Weyl anomaly, *Fortschr. Phys.* 48 (2000), 125–128 [arXiv:hep-th/9812032].
- [19] E. Kamke, *Differentialgleichungen. Lösungsmethoden und Lösungen*, Chelsea Publishing Company, 1971.
- [20] P. Kraus, F. Larsen and R. Siebelink, The gravitational action in asymptotically AdS and flat space-times, *Nucl. Phys. B* 563 (1999), 259–278 [arXiv:hep-th/9906127].
- [21] J. M. Maldacena, The large  $N$  limit of superconformal field theories and supergravity, *Adv. Theor. Math. Phys.* 2 (1998), 231–252 [*Internat. J. Theor. Phys.* 38 (1999), 1113–1133] [arXiv:hep-th/9711200].
- [22] D. Martelli and W. Muck, Holographic renormalization and Ward identities with the Hamilton–Jacobi method, *Nucl. Phys. B* 654 (2003), 248–276 [arXiv:hep-th/0205061].
- [23] H. Osborn and A. C. Petkou, Implications of conformal invariance in field theories for general dimensions, *Ann. Physics* 231 (1994), 311–362 [arXiv:hep-th/9307010].
- [24] I. Papadimitriou and K. Skenderis, Correlation functions in holographic RG flows, *J. High Energy Phys.* 0410 (2004), 075 [arXiv:hep-th/0407071].
- [25] A. Petkou and K. Skenderis, A non-renormalization theorem for conformal anomalies, *Nucl. Phys. B* 561 (1999), 100–116 [arXiv:hep-th/9906030].

- [26] K. Skenderis, Lecture notes on holographic renormalization, *Classical Quantum Gravity* 19 (2002), 5849–5876 [arXiv:hep-th/0209067].
- [27] K. Skenderis and S. N. Solodukhin, Quantum effective action from the AdS/CFT correspondence, *Phys. Lett. B* 472 (2000), 316–322 [arXiv:hep-th/9910023].
- [28] K. Skenderis and P. K. Townsend, Gravitational stability and renormalization-group flow, *Phys. Lett. B* 468 (1999), 46–51 [arXiv:hep-th/9909070].
- [29] P. K. Townsend, Positive energy and the scalar potential in higher-dimensional (super)gravity theories, *Phys. Lett. B* 148 (1984), 55–59.
- [30] R. M. Wald, *General Relativity*, The University of Chicago Press, Chicago 1984.
- [31] E. Witten, Anti-de Sitter space and holography, *Adv. Theor. Math. Phys.* 2 (1998), 253–291 [arXiv:hep-th/9802150].



# Mass formulae for asymptotically hyperbolic manifolds

Marc Herzlich

*Institut de Mathématiques et Modélisation de Montpellier  
CNRS – Université Montpellier II, 34095 Montpellier Cedex 5, France  
email: herzlich@math.univ-montp2.fr*

Among all possible aspects of AdS–CFT correspondence, there is one that is especially appealing for a mathematician: the ideas behind the correspondence provide indeed a strong link between conformal structures on compact manifolds and Einstein metrics on complete manifolds. The most striking example is of course the round conformal sphere, which can be efficiently described as the boundary at infinity of the unit ball with its hyperbolic metric, so that conformal geometry of the sphere and Riemannian geometry of hyperbolic space are intimately tied.

Whenever one wants to find conformal invariants of a compact manifold  $X$ , this example suggests to seek a manifold  $M$  whose boundary is  $X$  and a nice complete metric on the interior of  $M$  that “induces” the conformal structure on its “boundary at infinity”  $X$ . If it is chosen canonically enough, then the Riemannian invariants of  $M$  should reflect the conformal geometry of  $X$ . More precisely, given a conformal manifold  $(X, [\gamma])$ , what one looks for is a complete Einstein manifold  $(M, g)$  with negative scalar curvature, such that  $(X, [\gamma])$  stands at infinity of  $(M, g)$  in a manner similar to the sphere being the boundary at infinity of hyperbolic space.

In case one is interested in *local* invariants of  $[\gamma]$ , depending on a finite jet at a point, it is often (but not always) enough to have a power series expansion of  $g$  at infinity. Starting from the seminal work of Charles Fefferman and C. Robin Graham [13], this has been extensively used and has produced a lot of profound works in conformal geometry. However *non-local* invariants of  $X$  are expected to depend not only on a formal power series Einstein metric, but on the *global* geometry of  $M$ . As one faces the task to find a global Einstein metric  $g$  with adequate boundary-at-infinity conditions, hard existence and uniqueness problems then immediately arise.

We shall concentrate here on the latter. The first, too optimistic by large, question one could ask is the following: given two complete Einstein manifolds  $(M_1, g_1)$  and  $(M_2, g_2)$  that “have the same infinity” (providing a meaningful definition of this being part of the question), does one have that  $M_1 = M_2$  and  $g_1 = g_2$ ? Assuming that one of the manifolds is the hyperbolic space leads to an interesting sub-question, and answers do exist indeed in this case, as we shall see below. Going the other way round to the more general, one can also ask: given again  $(M_1, g_1)$  and  $(M_2, g_2)$  “looking



alike at infinity”, what kind of minimal geometric assumptions can one point out that ensure identity between them? Or better said: what quantities can we single out (e.g. compute) that make sure that they are genuinely different?

Depending on the context, many different answers can be given. If one sticks to AdS–CFT correspondence, one is entitled to stay within Einstein metrics, only the first questions matter, and uniqueness theorems for this case have been proven. But, broadening the perspective a little bit, one sees that the second general question exhibits close analogies with well-known aspects of what is called *asymptotically flat geometry*. Since this is one of the main mathematical motivations of what will be described below, we shall spend a few lines on it. An asymptotically flat manifold is a complete Riemannian manifold  $(M, g)$  such that the complement of a compact set is diffeomorphic to the complement of a ball in  $\mathbb{R}^n$  and such that the coefficients  $g_{ij}$  of the metric, when pulled back on  $\mathbb{R}^n$  by such a diffeomorphism (usually called a chart at infinity) converge to the euclidean metric coefficients  $\delta_{ij}$ . Although this may look a rather unnatural definition, this is a well-defined geometric notion [5], [4]. One of the most interesting aspects of such manifolds is the existence of a rather unique Riemannian invariant called *mass*, first defined in general relativity: this is an invariant computed in any chart at infinity as

$$m(g) = \lim_{r \rightarrow \infty} \int_{S(r)} (\operatorname{div} g - d \operatorname{tr} g)(\nu) d\mu_{S_r} \quad (0.1)$$

where  $\nu$  is the outer unit normal to the round sphere  $S_r$  of radius  $r$  in  $\mathbb{R}^n$ ,  $d\mu$  is its volume form, and divergence and trace refer to the euclidean metric. In case  $(g_{ij})$  converges quickly enough to  $(\delta_{ij})$ , this can be given a meaningful sense and does provide a Riemannian invariant of the asymptotically flat metric  $g$ . Even more strikingly, one has the well known *positive mass* phenomenon, proved by R. Schoen and S. T. Yau [25], [27] in dimensions less than  $8^1$  and by E. Witten [29] in any dimension for spin manifolds: if  $(M, g)$  is an asymptotically flat manifold with non-negative scalar curvature, and whose mass is well-defined, then  $m(g)$  is non-negative and is zero if and only if  $(M, g)$  is *isometric* to euclidean space.

The main goal of this text is to explain how both the concept of mass and the subsequent positivity and rigidity theorem can be generalized to the setting where the reference metric is no longer euclidean but hyperbolic. It is based on the various works done on the subject due to Lars Andersson and Mattias Dahl [3], Xiaodong Wang [28], Xiao Zhang [30], Piotr Chruściel and Gabriel Nagy [8] and Piotr Chruściel and the author [10].

Our journey will involve three steps. The first one is devoted to defining “asymptotically hyperbolic geometry”. Different definitions co-exist in the literature (usually under the same name!) and each of them raises specific difficulties, adding to the confusion of the unwarned reader. We shall consider two possible settings in this text, and for readers’ sake, we have tried to use different words for them: the first one will be called *conformally compact asymptotic hyperbolicity*, whereas we shall meet the sec-

<sup>1</sup>U. Christ and J. Lohkamp recently announced a proof in all dimensions.

and under the guise of *chart-dependent asymptotic hyperbolicity*. Of course, both notions are geometrically well-defined, as we shall show; for instance, we shall prove that, despite its name, chart-dependent asymptotic hyperbolicity is independent of any choice of chart! The second step is to define the mass and to check its consistency. Although formulated in different contexts, we shall show that each of the approaches described below lead to the same invariants. In a third step, we will conclude with some positivity and rigidity results (i.e. the relevant “positive mass theorem”). We have also tried to formulate a few open questions.

To conclude this introduction with an extra motivation for the reader who want to keep AdS–CFT correspondence in mind, let us mention that the work described below will lead us to the following rigidity theorem, which we shall state here in a slightly lousy manner (see Theorem 4.5 below for a precise statement):

**Theorem.** *Let  $M$  be a compact spin manifold with boundary  $X = \mathbb{S}^n$  (of dimension  $n \geq 3$ ). If  $g$  is a complete Einstein metric on the interior of  $M$  that induces the round conformal structure on  $\mathbb{S}^n$ , then  $M$  is topologically a ball and  $g$  is the hyperbolic metric.*

## 1 What is asymptotically hyperbolic geometry?

Two different paths can be followed to define asymptotically hyperbolic geometry. The first one is motivated by the links between hyperbolic space and the round conformal sphere. One could call it the “AdS–CFT procedure” or, as we shall do, “conformally compact geometry”. The second path is more in the spirit of asymptotically flat geometry: it defines model metric behaviours, and then describes what it means to be asymptotic to it in well-chosen charts.

### 1.1 The conformally compact approach

**1.1. Definition.** A Riemannian manifold  $(M, g)$  is  $C^k$  *conformally compact* if there exists a compact Riemannian manifold with boundary  $(\bar{M}, \bar{g})$  (with  $\bar{g}$   $C^k$  including on the boundary), together with a diffeomorphism  $\psi: \text{Int } \bar{M} \rightarrow M$  such that

$$\psi^* g = \Omega^{-2} \bar{g}, \quad (1.1)$$

where  $\Omega$  is a defining function for  $\partial \bar{M}$ , i.e.  $\Omega \geq 0$ ,  $\Omega^{-1}(0) = \partial \bar{M}$ , and  $d\Omega$  is nowhere vanishing on  $\partial \bar{M}$ .

Equation (1.1) determines only the conformal class  $[\bar{g}]$  of  $\bar{g}$  on  $\partial \bar{M}$ . Moreover, it is easily seen using the transformation properties of the Riemann tensor under conformal transformations (cf. e.g. [7]) that for smoothly  $C^2$  compactifiable metrics all the sectional curvatures  $\kappa$  of  $g$  satisfy

$$|d\Omega|_{\bar{g}}^2(p) + \kappa(p) \longrightarrow 0 \quad \text{as } p \rightarrow \partial \bar{M}, \quad (1.2)$$

where  $|\cdot|_k$  denotes the norm of a tensor with respect to a metric  $k$ .

A conformally compact manifold will be called *asymptotically hyperbolic in the conformally compact sense* if (assuming for convenience that it has only one end) it is conformally compact and one has

$$|d\Omega|_{\bar{g}}^2(p) = 1 \quad \text{for all } p \in \partial\bar{M}. \quad (1.3)$$

This covers a large class of manifolds, including of course hyperbolic space as well as a huge number of its quotients. It is easily shown (see [15], [20]) that, given a  $C^{k+1,\alpha}$ -compactifiable metric, one may choose a special defining function  $r$  (called a *geodesic defining function*) such that if  $\bar{g} = r^2 g$ ,

$$|dr|_{\bar{g}}^2 = 1 \quad (1.4)$$

in a collar neighbourhood of the boundary and not only on the boundary, as was the case in equation (1.3). In this case it can be shown that  $\bar{g}$  extends at least  $C^{k,\alpha}$  on the boundary and one may write the Taylor expansion

$$g = r^{-2}(dr^2 + h_0 + rh_1 + \cdots + r^k h_k + o(r^k)) \quad (1.5)$$

where all  $h_i$ 's are symmetric bilinear forms on the boundary  $\partial\bar{M}$ . The main advantage of this definition lies in the last expression (1.5) that is easy to manipulate as it translates any going-to-the-limit question into a problem at  $r = 0$ . As we shall see, its main problems are that it covers rather less examples than one would like to consider (especially if one is interested in General Relativity rather than in AdS–CFT correspondence) and that a fundamental issue is left aside: that of uniqueness of conformal completions.

## 1.2 The chart-dependent approach

Here we shall try to follow the path already known for asymptotically flat manifolds. The main idea is to define a limited number of model metrics together with adapted systems of frames, and to look at charts at infinity where metric coefficients converge towards the model metric coefficients.

If  $N$  is a compact manifold of dimension  $n - 1$ , we define the “model metric at infinity”  $b = b_k$  on  $N \times [R, +\infty[$  as

$$b_k = \frac{dr^2}{r^2 + k} + r^2 \check{h}, \quad (1.6)$$

with  $\check{h}$  a Riemannian metric on  $N$  with scalar curvature

$$\text{Scal}^{\check{h}} = (n - 1)(n - 2)k \quad \text{with } k \in \{0, \pm 1\}. \quad (1.7)$$

It is easily noticed that  $b_1$  is the hyperbolic metric if  $(N, \check{h})$  is the round sphere. In the same way if  $(N, \check{h})$  has constant curvature  $k$ , then  $b_k$  is of constant curvature  $-1$  as well. However, we insist on the fact that any  $\check{h}$  can be considered once the scalar curvature condition is satisfied (from the solution to the Yamabe problem [19], one

knows that this is not at all a demanding assumption as there is always such a metric in each conformal class).

We now label by  $\{f_i\}_{i=1,\dots,n}$  any (local)  $b_k$ -orthonormal frame on  $N \times [R, +\infty[$  of the following form:

$$f_i = r\epsilon_i, \quad i = 1 \dots n-1, \quad f_n = \sqrt{r^2 + k} \partial_r, \quad (1.8)$$

where the  $\epsilon_i$ 's form an orthonormal frame for the metric  $\check{h}$ . Metric coefficients will be expressed below in this choice of frame only.

**1.2. Definition.** A Riemannian manifold is *chart-dependent asymptotically hyperbolic of order  $\tau$*  if there exists a diffeomorphism  $\Phi: M \setminus K \rightarrow N \times [R, +\infty[$  (chart at infinity) such that

$$\sum_{i,j} |(\Phi_*g)_{ij} - \delta_{ij}| + \sum_{i,j,k} |f_k \cdot ((\Phi_*g)_{ij})| = O(r^{-\tau}) \quad (1.9)$$

for some  $\tau > 0$ .

Even if one restricts oneself to a small class of  $\check{h}$  metrics, this definition still covers a large class of asymptotic behaviours. Indeed, it might be impossible to attach any boundary at infinity (except in a very weak manner) to an asymptotically hyperbolic manifold in this sense, as Definition 1.2 allows for some quite wild behaviours. And using integral conditions instead of pointwise enables even wilder behaviours at infinity [10], [8].

Comparing with equation (1.5), one easily sees that a  $C^1$ -conformally compact asymptotically hyperbolic manifold is asymptotically hyperbolic in the sense of the last Definition 1.2, of order at least 1. However, we emphasize that an asymptotically hyperbolic manifold of order 1, say, has no reason to be even  $C^1$ -conformally compactifiable.

## 2 Uniqueness of structures at infinity

We shall here tackle the problems raised by both definitions of asymptotic hyperbolicity.

### 2.1 Uniqueness of conformal compactifications

In the conformally compact case, one should address the possible non-uniqueness of conformal compactifications whenever one is interested in defining invariants of the asymptotically hyperbolic metric. This discussion would of course be pointless if we were interested instead in constructing a conformally compact metric from a given

conformal manifold  $(X = \partial\bar{M}, [\gamma])$ . In general, if different conformal compactifications were to exist, they would yield two conformal manifolds that could not be told from another when seen from within  $M$ ; this seems to be very unlikely, and the next result rules out indeed this possibility, as expected.

**2.1. Theorem** (Chruściel–Herzlich [10]). *Let  $(M, g)$  be a conformally compactifiable asymptotically hyperbolic Riemannian manifold, that admits two  $C^\infty$ -conformal compactifications  $(\bar{M}_1 = M \cup \partial\bar{M}_1, \bar{g}_1, \Omega_1, \psi_1)$  and  $(\bar{M}_2 = M \cup \partial\bar{M}_2, \bar{g}_2, \Omega_2, \psi_2)$ . Then*

$$\Psi := \psi_1^{-1} \circ \psi_2 : \text{Int } M_2 \longrightarrow \text{Int } M_1 \quad (2.1)$$

*extends as a smooth conformal diffeomorphism from  $\bar{M}_2$  to  $\bar{M}_1$  in the sense of manifolds with boundary.*

For the reader's sake, we have stated the Theorem with strong  $C^\infty$  differentiability assumptions; the interested reader can however chase the order of differentiability in the proof of [10] to extend it to  $C^{k,\alpha}$  compactifications (presumably, the proof needs  $k \geq 2$ ).

We shall not give here the proof of Theorem 2.1; we shall only make a few comments. By using monotonicity properties of the geodesic defining functions (1.4), it is rather easy to show that  $\Psi$  extends as a bi-Lipschitz homeomorphism from  $\bar{M}_2$  to  $\bar{M}_1$ . Whenever this map is  $C^1$ , then one can use the deep theory of ACL<sup>n</sup> transformations due to Jacqueline Ferrand [21] to conclude that it is smooth and conformal in the strong sense. The only known proof of the  $C^1$ -differentiability is unfortunately (up to now) unsatisfactory, as it relies on the fact that any conformally compact asymptotically hyperbolic metric is asymptotically hyperbolic in the chart-dependent sense! The proof now uses the consistency results we shall describe in the next subsection to get further information on the map  $\Psi$  and eventually  $C^1$ -differentiability.

**2.2. Question.** Find a proof of this fact that is more in the spirit of the conformal geometric character of the problem.

## 2.2 Uniqueness for the chart-dependent asymptotically hyperbolic geometry

Recalling that our goal is to define invariants for a chart-dependent asymptotically hyperbolic metric one is entitled to expect that any two charts exhibiting convergence of the metric coefficients towards those of any reference metric will yield the same values for the invariants.

Let  $(M, g)$  be any chart-dependent asymptotically hyperbolic metric of order  $\tau > 0$ . From the definition, we have a chart at infinity

$$\Phi : M \setminus K \rightarrow N \times [R, +\infty[$$

such that, in the special frames already defined in the last section,

$$\sum_{i,j} |(\Phi_*g)_{ij} - \delta_{ij}| + \sum_{i,j,k} |f_k \cdot ((\Phi_*g)_{ij})| = O(r^{-\tau}). \quad (2.2)$$

As is easily seen, changing  $\Phi$  into  $A \circ \Phi$  where  $A$  is an isometry of the reference metric yields a new chart at infinity where decay assumptions (2.2) are satisfied too.

Hence, we have to face the following situation, where there might exist two charts at infinity

$$\begin{array}{ccc} & M \setminus K & \\ \swarrow \Phi_1 & & \searrow \Phi_2 \\ N \times [R, +\infty[ & & N \times [R, +\infty[ \end{array} \quad (2.3)$$

such that both metrics  $h_i = (\Phi_i)_*g$  ( $i = 1, 2$ ) satisfy condition (2.2) in adapted frame coefficients. The geometric well-definedness question will now be solved if the change of coordinates  $\Phi = \Phi_2 \circ \Phi_1^{-1}$  is close to an isometry of the background  $b$ . Said differently, this would ensure that for any chart-dependent asymptotically hyperbolic metric there exists a well-defined limiting model metric at infinity, and that all admissible charts at infinity are equivalent modulo isometries of the background. This question is tackled in full detail in [8], where the following is proved.

**2.3. Theorem** (Chruściel–Nagy [8]). *Let  $b = b_k$  be any of the model metrics defined above, with  $\check{h}$  of constant scalar curvature normalized by  $k = 1$ ,  $k = 0$  or  $k = -1$ . Let  $\Phi$  be any diffeomorphism of  $N \times [R, +\infty[$  satisfying*

$$\Phi^*b = b + O(r^{-\tau}) \quad (2.4)$$

*for some  $\tau > 0$ , and the same for its first derivatives. Then there exist an isometry  $A$  of  $b$  such that  $\Phi - A = O(r^{-\tau})$  and the same applies for first derivatives.*

Although this result may seem very natural, its proof is a long computational one, that we shall skip to keep a reasonable length to this report. It is based on two steps: in the first one shows that the coordinate change  $\Phi$  induces a conformal map of  $(N, \check{h})$ . In all cases we consider an explicit extension of this conformal map to an isometric map  $A$  of  $(N \times [R, +\infty[, b)$  can be given (see for instance the explicit proof given in [10, Prop. 5.2]), and in the second step one produces the desired estimates. The whole argument is quite involved and relies on very heavy computations; this is the reason why we shall say nothing more on this proof.

To give the reader an idea of an alternative route to this result, we will nevertheless sketch a much simpler conceptual proof in a special case, inspired by the elegant argument used by Robert Bartnik to prove the (analogously stated) consistency of asymptotically flat geometry [5].

**2.4. Proposition.** *Let  $k = 1$  and  $(N, \check{h})$  be the round sphere, so that  $b$  is the hyperbolic metric. If moreover the order of decay of  $h_i = (\Phi_i)_*g$  towards  $b$  is  $\tau > 2$ , then there*

exists an isometry  $A$  in  $O_0(n, 1)$  such that  $\Phi - A = O(r^{-\rho})$  in  $C^0$ -topology and for some  $\rho$  in  $]0, \tau[$ .

*Proof.* The idea is to find a finite dimensional space  $\mathcal{N}$  of functions whose definition does not involve a choice of chart at infinity, but chosen so that whenever a chart is specified, the elements of  $\mathcal{N}$  are asymptotic to a set of coordinate functions of the model space. Together with equation (2.4) this implies that both sets of coordinate functions (one for each choice of chart at infinity) correspond through an isometry of the model, and this in turn is a rephrasing of the conclusion of Theorem 2.3. It is very unfortunate that this proof works only when  $k = 1$  and the decay order is large.

Let us first recall that any model metric is  $b = (r + 1)^{-2}dr^2 + r^2\check{h}$  on  $\mathbb{S}^{n-1} \times [R, +\infty[$ . For an asymptotically hyperbolic metric  $g$  we now define  $H(U) = \text{Hess}^g U - Ug$  and  $D(U) = \Delta^g U + nU$  for any function  $U$  in  $M$ . Setting  $\tau = 1 + \delta$  we let

$$\mathcal{N} = \{U \in C_{\text{loc}}^2 \cap C_{-1} \mid H(U) \in C_{1+\delta}, D(U) = 0\}$$

where  $C_\rho$  is the space of continuous functions such that  $(r \circ \Phi)^\rho u$  is bounded for any choice of  $\Phi$ . As all reference metrics are mutually bounded, *this definition does not depend* on the choice of chart  $\Phi$ . Hence  $\mathcal{N}$  is a well defined space for the metric  $g$ .

Let us choose for a moment a chart at infinity. For simplicity's sake, it is now convenient to use the new variable  $s$  defined by  $r = \sinh(s)$ ; this has the effect that the model metric is now  $b = ds^2 + \sinh^2(s)\check{h}$ . Then for any  $U$  in  $\mathcal{N}$  it is easy (although slightly tedious) to show that there exists continuous functions  $u_0$  and  $u_1$  on  $\mathbb{S}^{n-1}$  such that

$$U - \cosh(s)u_0 - \sinh(s)u_1 \in C_{1+\delta'} \quad (2.5)$$

for any  $\delta'$  in  $]0, \delta]$ . Moreover, this estimate extends to one radial (not tangential!) derivative. This is obtained in a rather standard way by treating the condition  $H(U) \in C_{1+\delta}$  as an ordinary differential equation along any radial geodesic ray in the model metric, and it is here that we require  $\delta > 1$  as such a decay is needed to get the precise asymptotic estimate (2.5).

Restricting now the information on  $H(U)$  to the tangent spaces of the spheres  $\mathbb{S}^{n-1} \times \{r\}$  one gets that

$$\text{Hess}^{\sinh^2(s)\check{h}} U = -\sinh(s)u_1\check{h} + \text{pert.},$$

where “pert.” denotes a perturbation term living in  $C_{\delta''}$  for some  $\delta'' > 0$ . Computing the (smooth) combinations

$$\cosh(s)\partial_s U - \sinh(s)U \quad \text{and} \quad \cosh(s)U - \sinh(s)\partial_s U$$

and recalling that estimates (2.5) extend to one radial derivative, we end up with

$$\text{Hess}^{\check{h}}(u_0 + \text{pert.}) = \text{pert.} \quad \text{and} \quad \text{Hess}^{\check{h}}(u_1 + \text{pert.}) = -u_1 + \text{pert.}$$

(same convention as above for the perturbation terms; this means here that each perturbation term is  $O(e^{-\varepsilon s})$  for some  $\varepsilon > 0$ ). Integrating the previous formulas by parts against any smooth function on the sphere and letting  $s$  go to infinity shows that  $u_0$  and  $u_1$  are (weak) solutions of the equations  $\Delta^{\check{h}} u_0 = 0$  and  $\Delta^{\check{h}} u_1 = n u_1$ .

These functions are of course well-known as are the corresponding functions  $\cosh(s)u_0 + \sinh(s)u_1$  on hyperbolic space: they form the finite dimensional space

$$\mathcal{N}_b = \{V \in C^\infty \mid \text{Hess}^b V = Vb\}. \quad (2.6)$$

Whenever hyperbolic space is seen as the upper hyperboloid in Minkowski space  $\mathbb{R}^{n+1}$ ,  $\mathcal{N}_b$  is nothing else but  $\mathbb{R}^{n+1}$  itself (seen as the space of the linear coordinate functions restricted to the hyperboloid).

Our result so far is that for any  $U$  in  $\mathcal{N}$  and any choice of chart at infinity, there exists  $V$  in  $\mathcal{N}_b$  such that (relatively to the asymptotic chart  $\Phi$ )

$$U - V \circ \Phi \in C_{1+\delta'}.$$

From this, standard arguments of analysis applied to  $D = \Delta + n$  on asymptotically hyperbolic spaces (see e.g. [2], [23]) show that  $\mathcal{N}$  is finite dimensional and in one-to-one correspondence with  $\mathcal{N}_b$ .

One may now conclude the proof: given charts at infinity  $\Phi_1$  and  $\Phi_2$  satisfying conditions (2.2), we get from the previous analysis a matrix  $A$  in  $\text{GL}(n+1, \mathbb{R})$  such that

$$V^{(i)} \circ \Phi_2 \circ \Phi_1^{-1} - \sum_j A_j^i V^{(j)} \in C_{\delta'} \quad \text{for all } i, j$$

where  $\Phi = \Phi_2 \circ \Phi_1^{-1}$  is seen as a diffeomorphism of the hyperboloid and  $\{V^{(\alpha)}\}$  denote any Minkowski orthonormal basis of  $\mathcal{N}_b$ . The previous formula shows that  $\Phi$  acts as  $A$  at infinity in hyperbolic space and is there an isometry. Hence  $A$  belongs to  $O_0(n, 1)$ . Here again we have to use the fact that  $\delta > 1$ : indeed a lower decay would not yield enough information from the equation

$$|V^{(0)}|^2 - \sum_{\alpha=1}^n |V^{(\alpha)}|^2 = 1.$$

Recalling that  $(V^{(\alpha)})$  forms a full set of coordinates for the standard hyperboloidal embedding of hyperbolic space shows that  $(\Phi - A)$  must be an element of  $C_{\delta''}$ , as expected.  $\square$

**2.5. Question.** It is an interesting question whether the “asymptotic rigidity” of isometries in hyperbolic metrics is a general fact or not, i.e. whether any diffeomorphism of a complete Riemannian manifold  $(M, b)$  ( $b$  arbitrary here) satisfying (2.4) is necessarily close to an isometry. Piotr Chruściel and Gabriel Nagy extend in [8] this property to a larger set of metrics including the ones considered in this survey. The key element is always to consider metrics such that any conformal map at infinity can



be extended to an isometry in the bulk manifold. Following this idea, such a rigidity property can also be shown to hold true in all non-compact rank one symmetric spaces [16].

### 3 Invariants at infinity for asymptotically hyperbolic metrics

We can now achieve our goal, that is to say defining an adequate notion of mass for asymptotically hyperbolic manifolds. As always, we shall consider separately our two settings of asymptotic conditions. In both cases, we shall give conditions that ensure existence of the invariant, and explain why it is well defined. In contrast with the asymptotically flat case, the most striking feature we shall meet here is that mass is not a number but a function on a well-chosen space.

#### 3.1 Conformally compact manifolds approach

We follow here the ideas introduced by Xiaodong Wang [28].

*Decay conditions for existence of mass.* Let  $(M^n, g)$  be a  $C^n$ -conformally compact asymptotically hyperbolic manifold such that its conformal boundary is the standard round sphere  $(\mathbb{S}^{n-1}, [\check{h}])$ . We will furthermore assume that the metric  $g$  admits the following asymptotic expansion

$$g = \sinh(r)^{-2} \left( dr^2 + \check{h} + \frac{r^n}{n!} h + o(r^{n+1}) \right) \quad (3.1)$$

for some defining function  $r$  of the boundary at infinity (this is not a geodesic defining function as previously defined, but existence of this one follows from completely analogous arguments).

**3.1. Definition (Wang).** The mass of the asymptotically hyperbolic manifold  $(M, g)$  is the vector

$$\left( \int_{\mathbb{S}^{n-1}} \text{tr}_{\check{h}} h d\mu_{\check{h}}, \int_{\mathbb{S}^{n-1}} x^1 \text{tr}_{\check{h}} h d\mu_{\check{h}}, \dots, \int_{\mathbb{S}^{n-1}} x^n \text{tr}_{\check{h}} h d\mu_{\check{h}} \right) \in \mathbb{R}^{n+1}, \quad (3.2)$$

where  $x^1, \dots, x^n$  are the standard linear coordinate functions on  $\mathbb{S}^{n-1}$ .

As will be apparent when comparing to the chart-dependent definition of mass, it is perhaps more convenient to see the mass as a linear map  $W$  on  $\mathbb{R}^{n+1}$  rather than a vector, the map defined by

$$w = (w_0, \dots, w_n) \in \mathbb{R}^{n+1} \mapsto W(w) = \sum_{\alpha=0}^n \int_{\mathbb{S}^{n-1}} x^\alpha w_\alpha \text{tr}_{\check{h}} h d\mu_{\check{h}}, \quad (3.3)$$

with the convention that  $x^0$  is the constant function with value 1. It is important to notice at this point that  $W$  depends a priori on an explicit identification of the boundary at infinity to some constant curvature representative  $(\mathbb{S}^{n-1}, \check{h})$  of the round conformal structure.

From the definition, it is not clear at all that this stands as a generalization of the mass to the asymptotically hyperbolic context. The justification comes either from the relevant positive mass theorem which will be described in Section 4 or from the comparison with the chart-dependent approach that will show that there does exist an alternative expression for the mass that is more in spirit of that in the asymptotically flat case.

*Geometric consistency.* As Theorem 2.1 ensures uniqueness of conformal completions, consistency is obtained through the following result, whose proof consists in rather lengthy computations.

**3.2. Lemma** (Wang [28]). *Let  $(M, g)$  be a conformally compact asymptotically hyperbolic manifold satisfying condition (3.1), and let  $\hat{h}$  be another round metric on  $\mathbb{S}^{n-1}$ . Then  $g$  satisfies also condition (3.1) for the relevant defining function. Moreover, if  $A$  is the conformal map in  $O_0(n, 1)$  such that  $\hat{h} = A^* \check{h}$ , then*

$$W_{\hat{h}} = W_{\check{h}} \circ A.$$

### 3.2 Chart-dependent approach

*Decay conditions.* Let  $(M, g)$  be an asymptotically hyperbolic manifold, i.e. there exists a chart at infinity  $\Phi$  such that

$$\sum_{i,j} |(\Phi_* g)_{ij} - \delta_{ij}| + \sum_{i,j,k} |f_k \cdot ((\Phi_* g)_{ij})| = O(r^{-\tau}). \quad (3.4)$$

We further assume here that  $\tau > \frac{n}{2}$  and that the difference between the scalar curvature of  $g$  and that of  $b$  is in  $L^1$ .

We let  $e = (\Phi_* g) - b$  on  $N \times [R, +\infty[$  and we define for  $b = b_k$ :

$$\mathcal{N}_b = \{V \in C^\infty(N \times [R, +\infty[) \mid \text{Hess}^b V = V(\text{Ric}^b + nb)\}. \quad (3.5)$$

Contrarily to the conformally compact case, we here insist on the fact that we make no further assumptions on  $N$ , except those restrictions imposed from the beginning to the metric  $\check{h}$ . The most important special cases are when the following assumptions are satisfied.

*The metric  $\check{h}$  is either of constant curvature  $k = 0$  or  $k = 1$ , or it has negative Ricci curvature and in this case one sets  $k = -1$ .*

The reason for considering primarily these cases is that the structure of the space  $\mathcal{N}_b$  is well-known there: it is easily shown that, if  $k \leq 0$ ,  $\mathcal{N}_b$  is 1-dimensional,

generated by  $V^{(0)} = \sqrt{r^2 + k}$ , whereas, if  $k = 1$  and if the reference metric is the hyperbolic metric,  $\mathcal{N}_b$  has already been described: it is  $n + 1$ -dimensional and consists of the linear coordinates of  $\mathbb{R}^{n+1}$  seen as functions on the standard hyperboloid. Note moreover that, irrespective of any assumption on  $\check{h}$ ,  $\mathcal{N}_b$  is always finite-dimensional, of dimension bounded above by  $(n + 1)$ .

**3.3. Definition.** The mass of  $(M, g)$  (relative to the chart  $\Phi$ ) is the linear functional  $H_\Phi$  on  $\mathcal{N}_b$  defined by

$$H_\Phi(V) = \lim_{r \rightarrow \infty} \int_{S_r} (V(\operatorname{div}^b e - d \operatorname{tr}_b e)(v_r) + (\operatorname{tr}_b e) dV(v_r) - e(\nabla^b V, v_r)) d\mu, \quad (3.6)$$

where  $S_r = N \times \{r\}$ ,  $v_r$  is its unit outer normal and  $d\mu$  is its volume form induced by  $b$ .

Comparing with (0.1), it is now clear that, whenever we are able to prove its convergence and consistency, any such an  $H_\Phi$  is a strong candidate to be a generalization of mass to the setting of asymptotically hyperbolic manifolds.

We prove the convergence of the limit by the same argument that already applies in the asymptotically flat context. In a sense, this argument might be seen as one of the main reasons for the particular expression under the integral sign. It is based on the following first variation formula [7, chapter I] :

$$\frac{d}{dt} (\operatorname{Scal}^{b+te})|_{t=0} = \operatorname{div}^b \operatorname{div}^b e + \Delta^b \operatorname{tr}_b e - \langle e, \operatorname{Ric}^b \rangle \quad (3.7)$$

so that for any  $V$  in  $\mathcal{N}_b$ ,

$$\begin{aligned} V \frac{d}{dt} (\operatorname{Scal}^{b+te})|_{t=0} &= V(\operatorname{div}^b \operatorname{div}^b e + \Delta^b \operatorname{tr}_b e) - \langle e, \operatorname{Hess}^b V + (\Delta^b V)b \rangle \\ &= \operatorname{div}^b (V \operatorname{div}^b e - V d \operatorname{tr}_b e - e(\nabla^b V, \cdot) + (\operatorname{tr}_b e) dV). \end{aligned} \quad (3.8)$$

Thus, the expression of  $H_\Phi(V)$  can be transformed into a bulk integral

$$\int_{N \times [R, r]} (V(\operatorname{Scal}^{b+e} - \operatorname{Scal}^b) + \mathcal{Q}(V, e))$$

where  $\mathcal{Q}(V, e)$  is a polynomial expression that is linear in  $V$  and quadratic in  $e$  and its first derivatives. Recalling that any element  $V$  in  $\mathcal{N}_b$  has growth  $V = O(r)$ , decay conditions (3.4) are now fit to prove that the bulk integral converges, thus implying the convergence of the mass expression for any  $V$  in  $\mathcal{N}_b$ .

Of course, in the special cases singled out above, one can be more precise: if  $k \leq 0$  one has  $\dim \mathcal{N}_b = 1$  and seeing the mass as a linear functional is just a complicated way of getting a number. However, it is important to keep this interpretation in mind in the case  $k = 1$ , as it will play an important role when we will consider the behaviour of mass under coordinate changes.

*Connection with the previous approach.* In the case that  $(N, \check{h})$  is the canonical round sphere  $\mathbb{S}^{n-1}$  and  $k = 1$  ( $b$  is then the hyperbolic metric), it is a simple calculation to show that the mass defined in this way completely supersedes the previous definition for conformally compact manifolds (although the latter might be of much simpler use in practical cases). To make the comparison easier with the previous approach, it is of course useful to see the conformally compact definition of mass as a linear functional on the space  $\mathbb{R}^{n+1}$ , as we already did in formula (3.3). Going along the reverse path, one may also choose to see the chart-dependent mass as a function on isotropic directions in  $\mathbb{R}^{n+1}$ , or, said in a different manner, on the boundary at infinity  $\mathbb{S}^{n-1}$  of hyperbolic space.

*Invariance with respect to coordinate transformations.* The well-definedness property of the mass is less obvious from its definition. As in the euclidean case, it relies on a clever computation, and, as Robert Bartnik once called it, a “miraculous cancellation” [5].

The idea is of course to use the geometric consistency Theorem 2.3 for asymptotically hyperbolic geometry; we shall here give only a sketch and we refer to [10] for details. It is enough to consider the case where one has two charts  $\Phi_1$  and  $\Phi_2$  such that  $\Phi_2 \circ \Phi_1^{-1} = \text{Id} + O(r^{-\tau})$ . If  $\mathbb{U}_i(V)$  ( $i = 1, 2$ ) are the integrands of the mass expressions in both charts, one is bound to compute

$$\delta\mathbb{U} = \mathbb{U}_2(V) - \mathbb{U}_1(V)$$

The computation then proceed as follows: one expresses how each of the quantities in the integrand varies when passing from the first chart to the second. One then collects all terms of decay order larger than  $n - 1$ , so that

$$\delta\mathbb{U} = \delta\mathbb{U}(1) + o(r^{1-n})$$

where  $\delta\mathbb{U}(1)$  is a “first-order” term whose decay is a priori too weak to have limit zero when integrated on  $N \times \{r\}$  when  $r$  goes to infinity. But here comes the “miraculous cancellation”, that is the fact that one may modify  $\delta\mathbb{U}(1)$  by performing integration by parts, and due to the special choice of  $\mathcal{N}_b$  one ends up with

$$\delta\mathbb{U}(1) = \text{divergence term} + o(r^{1-n}) !$$

Invariance of the mass follows immediately, leading to the following statement.

**3.4. Theorem.** *Let  $\Phi_2$  and  $\Phi_1$  two charts at infinity as in Theorem 2.3, so that  $\Phi_2 \circ \Phi_1^{-1} = A + O(r^{-\tau})$  for some  $A$  in the isometry group of  $b$ , and the same for derivatives. Then*

$$H_{\Phi_2}(V) = H_{\Phi_1}(V \circ A^{-1})$$

for any element  $V$  of  $\mathcal{N}_b$ .

**3.5. Question.** The miraculous cancellation alluded to above is a phenomenon that has not yet found a definitive explanation. It seems to be much related to diffeomorphism

invariance of the scalar curvature and of the first variation formula (3.8) but we are still lacking of a geometric interpretation.

The reader might of course wonder what is the role of the space of functions  $\mathcal{N}_b$  and why the mass should be a linear functional, and not a number as in the asymptotically flat case. An explanation can be found in General Relativity, which we shall sketch below, referring to [6] for computations in the asymptotically flat case and to [11] for a thorough study of the whole story.

The key idea is that the mass can be seen as an adequate Hamiltonian function for the description of the Einstein equations for a Lorentz metric  $\gamma$  on an  $(n+1)$ -dimensional manifold  $P$  as a Hamiltonian system, at least at a *formal* level (we do not claim here to describe any well-founded theory of Hamiltonian systems in infinite dimensions adapted to General Relativity). To effectively perform this formal description, it is necessary to write the equations as a first-order system. One classical way to do that is to assume the existence of a reference metric  $\beta$  in  $P$  (e.g. the Minkowski or anti-de Sitter metric). One then fixes an initial slice  $M$  in  $P$  and an identification of a neighbourhood of  $M$  with  $M \times ]-\varepsilon, \varepsilon[$ , the last factor being timelike. Very roughly speaking, one now considers as unknowns the difference between the metric  $g$  induced by  $\gamma$  on  $M \times \{t\}$  and the one (denoted by  $b$ ) induced by the reference metric, and the difference between the corresponding second fundamental forms.

If one now wishes to work within the class of asymptotically Minkowski (or Anti-deSitter) metrics, it is reasonable to take as  $M$  a constant curvature (0 or  $-1$ , depending on the case of interest) Riemannian hypersurface in  $(P, \beta)$  and to assume that each  $M \times \{t\}$  is given by the flow of a Killing vector field  $X$  of  $\beta$  (if one wishes to stay in the realm of metrics asymptotic to the reference one, this is obviously something one should do!).

If the Killing field  $X$  is hypersurface orthogonal in  $\beta$ , the mass can then be taken as a Hamiltonian function for the equations.<sup>2</sup> But one gets of course one Hamiltonian function for each choice of Killing field, and if two choices are related by an isometry of  $\beta$  preserving (globally, not pointwise) the initial slice, then the corresponding Hamiltonians should be related. As a first try, set  $P = (\mathbb{R}^{n+1}, \text{Minkowski})$ ,  $M = (\mathbb{R}^n, \text{Eucl.})$  and  $X = \partial_t$ ; then this yields the asymptotically euclidean mass (and it is unique as the time translation is fixed by the action of the isometry group of  $M$  seen as acting on  $P$ ). Setting now  $P$  to be the Anti-deSitter space and  $M$  as the hyperbolic space enables more freedom, as there are a large space of hypersurface-orthogonal Killing vectors, precisely those that can be written as an element of  $\mathcal{N}_b$  times the  $\beta$ -unit normal vector of  $M$  in  $P$ . This is an  $(n+1)$ -dimensional space, and the action of the group  $O_0(n, 1)$  of (time-oriented) isometries of the hyperbolic space is through its standard representation. This explains why the mass is a function on the space  $\mathcal{N}_b$ .

---

<sup>2</sup>This is not quite true: Einstein equations split in two subsystems, the first one formed of “evolution” equations, whereas the second one called “constraints” applies to the initial data; mass is then (formally) a Hamiltonian for the evolution equations when one restricts it to metrics and second forms in the solution space of the constraints equations.

This argument also shows that the story can be continued one step further, by considering embeddings of Riemannian manifolds that closely resemble the standard embedding of hyperbolic space into the Anti-deSitter space. One may define there a larger set of mass (or rather “energy-momentum”) invariants, depending not only on the difference between the metric and the model hyperbolic metric, but also on second fundamental forms. For extra information on this point we refer to [8].

## 4 Rigidity theorems

We are now ready to prove our main rigidity theorems. As already noticed, one of the most important results on the mass of asymptotically flat geometry is the following celebrated “positive mass theorem” of Schoen–Yau [25], [27]<sup>3</sup> and Witten [29]:

**4.1. Theorem** (Christ–Lohkamp, Schoen–Yau, Witten). *Let  $(M, g)$  be a Riemannian asymptotically flat manifold with non-negative scalar curvature. Then the mass of  $M$ , if it exists, is non-negative, and is zero if and only if  $(M, g)$  is isometric to euclidean space.*

In the asymptotically hyperbolic setting this theorem has a very close analog that we now state [10], [28]:

**4.2. Theorem** (Chruściel–Herzlich, Wang). *Let  $(M, g)$  be an asymptotically hyperbolic spin manifold, either in the conformally compact sense with boundary at infinity  $(\mathbb{S}^{n-1}, [h])$  or in the chart dependent sense with  $k = +1$ . Assume moreover that the scalar curvature of  $g$  is no less than  $-n(n-1)$ , and that decay conditions are satisfied so that the mass exists. Then the mass linear functional is either timelike future directed or zero, and this last case can happen only if  $(M, g)$  is isometric to hyperbolic space.*

Recall that a vector  $v$  in the time-oriented Minkowski space  $(\mathbb{R}^{n+1}, \langle \cdot, \cdot \rangle)$  is said to be timelike if  $\langle v, v \rangle < 0$  and future (resp. past) directed if it can be written  $v = (v_0, v_1, \dots, v_n)$  with  $v_0 > 0$  (resp.  $v_0 < 0$ ) in a time-oriented orthonormal basis  $e_0, \dots, e_n$  of  $\mathbb{R}^{n+1}$  with  $\langle e_0, e_0 \rangle = -1$ . A co-vector is then said to be timelike future-directed (resp. past-directed) if the dual vector is timelike past-directed (resp. future directed).

The proof of the last theorem mainly follows Witten’s idea for the positive mass theorem. The key idea is that mass appears as the boundary contribution in an integration by parts formula involving the Dirac operator on spinors (hence the spin assumption). By choosing adequately the spinor fields under consideration, non-negativity is obtained because the bulk integrals are obviously non-negative. Rigidity follows from

---

<sup>3</sup>See footnote 1, p. 104.

the fact that mass being zero implies that some spinor field satisfied a strong overdetermined equation. Detailed proofs can be found in [10] and [28]. Note moreover that Xiaodong Wang indeed extracted his definition of mass from the spinorial Stokes' formula.

**4.3. Remark.** The positive mass theorem can be extended to include the “extrinsic” energy-momentum invariants depending on the metric and a second fundamental form already alluded to above. This is well-known in the asymptotically flat setting, see for instance [24], [26], [29], and is currently under scrutiny for the asymptotically hyperbolic case ([12], [22]).

**4.4. Question.** There exist alternative proofs of the positive mass theorem in the asymptotically flat setting, relying on minimal surface considerations [25] or on inverse mean curvature flow [14], [18], [17]. It would be highly desirable to see whether these approaches could be of any use in this case.

*Application to the AdS–CFT correspondence and conformally compact Einstein metrics.* We now let  $(M, g)$  to be an Einstein  $C^2$ -conformally compact manifold, whose boundary at infinity is the round sphere  $(\mathbb{S}^{n-1}, [\check{h}])$ . Boundary expansions and regularity of conformally compact Einstein metrics of class at least  $C^2$  have been studied by C. Robin Graham [15], by Michael Anderson in dimension 4 [1] and by Piotr Chruściel, Erwann Delay, John M. Lee and Dale Skinner in all other dimensions [9].

In dimension 3 and all even dimensions, one gets that, in a well-chosen set of coordinates around the boundary at infinity  $[0, \varepsilon[\times\partial\bar{M}$ ,

$$g = \rho^{-2} (d\rho^2 + G(\rho))$$

with  $G$  a smooth function in  $\rho$ , that can be Taylor expanded as

$$G(\rho) = G_{(0)} + \rho^2 G_{(2)} + \cdots + G_{(n-2)} \rho^{n-2} + G_{(n-1)} \rho^{n-1} + o(\rho^{n-1}),$$

where the  $G_{(j)}$ 's are smooth symmetric bilinear forms on  $\partial\bar{M}$  that are formally (and locally) determined from the data on the boundary  $G_{(0)}$  for  $j \leq n-2$ , whereas  $G_{(n-1)}$  is trace-free and formally undetermined. This phenomenon is analogous to the indeterminacy of the Neumann data  $\partial_\nu \phi$  when one seeks a harmonic function  $\phi$  in a bounded set  $\Omega$  with boundary condition  $\phi = f$  on  $\partial\Omega$ : the Neumann data can be determined only from global considerations in  $\Omega$  and *not* from local Taylor expansions around  $\partial\Omega$ .

In odd dimensions  $n$  larger than 5, the compactified metric is only polyhomogeneous, in the sense that it can be written

$$g = \rho^{-2} (d\rho^2 + G(\rho, \rho^{n-1} \log \rho)),$$

in some well-chosen geodesic defining function, with  $G(x, z)$  a function in two variables (and the extra boundary variables) *smooth in all its arguments*. In this case, this

leads to an asymptotic expansion for the metric of the form

$$G(\rho, \rho^{n-1} \log \rho) = G_{(0)} + \rho^2 G_{(2)} + \cdots + G_{(n-3)} \rho^{n-3} \\ + G'_{(n-1)} \rho^{n-1} \log \rho + G_{(n-1)} \rho^{n-1} + o(\rho^{n-1}),$$

where again the  $G_{(j)}$ 's are smooth symmetric bilinear forms on  $\partial \bar{M}$  that are formally and locally determined from the data  $G_{(0)}$  for  $j \leq n-3$ , as is the trace of  $G_{(n-1)}$  and the new term  $G'_{(n-1)}$ . However, the trace-free part of  $G_{(n-1)}$  is formally undetermined.

In case the boundary at infinity is the round sphere  $(\mathbb{S}^{n-1}, [\check{h}])$ , the expansion is the same as the one of the hyperbolic metric up to order  $n-2$ . Moreover in odd dimension the extra term  $G'_{(n-1)}$  and the trace of  $G_{(n-1)}$  are formally determined, hence have the same value as in the hyperbolic metric (e.g.  $G'_{(n-1)} = 0$ ). This expansion ensures that the mass is well defined: for instance, the decay conditions needed for existence of the mass in the chart-dependent approach are easily seen to be met. Using this expansion, one can then show better: computations done by Lars Andersson and Mattias Dahl [3] show that the term  $\text{tr } G(\rho)$  (resp.  $\text{tr } G(\rho, \rho^n \ln \rho)$ ) is the same as the one in the hyperbolic metric up to a much higher order than  $n$ . As the mass depends only on the trace of the  $n$ -th order term, something that is immediately apparent from Wang's definition (3.2), this shows that the mass (whichever definition is used) is zero. Applying the positive mass rigidity theorems, one concludes:

**4.5. Theorem.** *Let  $(M, g)$  be a  $C^2$  conformally compact Einstein and spin manifold whose boundary at infinity is the round sphere  $\mathbb{S}^{n-1}, [\check{h}]$ . Then  $M$  is necessarily isometric to hyperbolic space.*

**Acknowledgements.** The author is grateful to J. Lafontaine and D. Maerten for useful comments on a first version of this text.

## References

- [1] M. T. Anderson, Boundary-regularity, uniqueness and non-uniqueness for AH Einstein metrics on 4-manifolds, *Adv. Math.* 179 (2003), 205–249.
- [2] L. Andersson, Elliptic systems on manifolds with asymptotically negative curvature, *Indiana Univ. Math. J.* 42 (1993), 1359–1388.
- [3] L. Andersson and M. Dahl, Scalar curvature rigidity for asymptotically locally hyperbolic manifolds, *Ann. Global Anal. Geom.* 16 (1998), 1–27.
- [4] S. Bando, A. Kasue, and H. Nakajima, On a construction of coordinates at infinity on manifolds with fast curvature decay and maximal volume growth, *Invent. Math.* 97 (1989), 313–349.
- [5] R. Bartnik, The mass of an asymptotically flat manifold, *Comm. Pure Appl. Math.* 39 (1986), 661–693.



- [6] —, Hamiltonian phase space for asymptotically flat data, unpublished manuscript, available on his web page <http://gular.ise.canberra.edu.au/~bartnik>, 1996.
- [7] A. L. Besse, *Einstein manifolds*, Ergeb. Math. Grenzgeb. (3) 10, Springer-Verlag, Berlin 1987.
- [8] P. Chruściel and G. Nagy, The mass of asymptotically anti-de Sitter space-times, *Adv. Theor. Math. Phys.* 5 (2001), 697–754.
- [9] P. T. Chruściel, E. Delay, J. M. Lee and D. Skinner, Boundary regularity of conformally compact Einstein metrics, preprint, 2004.
- [10] P. T. Chruściel and M. Herzlich, The mass of asymptotically hyperbolic Riemannian manifolds, *Pacific J. Math.* 212 (2003), 231–264.
- [11] P. T. Chruściel, J. Jezierski and J. Kijowski, *Hamiltonian field theory in the radiating regime*, Lecture Notes in Phys. 70, Springer-Verlag, Berlin 2001.
- [12] P. T. Chruściel, J. Jezierski, and S. Łęski, The Trautman–Bondi mass of initial data sets, *Adv. Theor. Math. Phys.* 8 (2004), 83–139.
- [13] C. Fefferman and C. R. Graham, Conformal invariants, in: *Élie Cartan et les mathématiques d'aujourd'hui* (Lyon 1984), *Astérisque* (1985), Numéro hors-série, 95–116.
- [14] R. Geroch, Energy extraction, *Ann. New York Acad. Sci.* 224 (1973), 108–117.
- [15] C. R. Graham, Volume and area renormalizations for conformally compact Einstein metrics, *Rend. Circ. Mat. Palermo*, Suppl. no. 63, 2000, 31–42.
- [16] M. Herzlich, Isometry rigidity for non-compact rank-one symmetric spaces, unpublished.
- [17] G. Huisken and T. Ilmanen, The inverse mean curvature flow and the Riemannian Penrose inequality, *J. Differential Geom.* 59 (2001), 353–437.
- [18] P. S. Jang, On the positive mass conjecture, *J. Math. Phys.* 17 (1976), 141–145.
- [19] J. Lee and T. H. Parker, The Yamabe problem, *Bull. Amer. Math. Soc.* 17 (1987), 37–91.
- [20] J. M. Lee, The spectrum of an asymptotically hyperbolic Einstein manifold, *Comm. Anal. Geom.* 3 (1995), 253–271.
- [21] J. Lelong-Ferrand, Geometrical interpretations of scalar curvature and regularity of conformal homeomorphisms, in: *Differential geometry and relativity*, Mathematical Phys. and Appl. Math. 3, Reidel, Dordrecht 1976, 91–105.
- [22] D. Maerten, Ph.D. thesis, Univ. Montpellier II, in preparation.
- [23] R. Mazzeo, The Hodge cohomology of a conformally compact manifold, *J. Differential Geom.* 28 (1988), 309–339.
- [24] T. H. Parker and C. H. Taubes, On Witten's proof of the positive energy theorem, *Comm. Math. Phys.* 84 (1982), 223–238.
- [25] R. Schoen and S. T. Yau, On the proof of the positive mass conjecture in General Relativity, *Comm. Math. Phys.* 65 (1979), 45–76.
- [26] —, The energy and linear-momentum of spacetimes in general relativity, *Comm. Math. Phys.* 79 (1981), 47–51.

- [27] —, Proof of the positive mass theorem II, *Comm. Math. Phys.* 79 (1981), 231–260.
- [28] X. Wang, Mass for asymptotically hyperbolic manifolds, *J. Differential Geom.* 57 (2001), 273–299.
- [29] E. Witten, A new proof of the positive energy theorem, *Comm. Math. Phys.* 80 (1981), 381–402.
- [30] X. Zhang, A definition of total energy-momenta and the positive mass theorem on asymptotically hyperbolic 3-manifolds, I, *Comm. Math. Phys.* 249 (2004), 529–548.



# Reconstructing Minkowski space-time

*Sergey N. Solodukhin*

*School of Engineering and Science, International University Bremen*

*P.O. Box 750561, Bremen 28759, Germany*

*email: s.solodukhin@iu-bremen.de*

**Abstract.** Minkowski space is a physically important space-time for which it is an urgent problem to find an adequate holographic description. In this paper we develop further the proposal made in [10] for the description as a duality between Minkowski space-time and a Conformal Field Theory defined on the boundary of the light-cone. We focus on the gravitational aspects of the duality. Specifically, we identify the gravitational holographic data and provide the way Minkowski space-time (understood in a more general context as a Ricci-flat space) is reconstructed from the data. In order to avoid the complexity of non-linear Einstein equations we consider linear perturbations and do the analysis for the perturbations. The analysis proceeds in two steps. We first reduce the problem in Minkowski space to an infinite set of field equations on de Sitter space one dimension lower. These equations are quite remarkable: they describe massless and massive gravitons in de Sitter space. In particular, the partially massless graviton appears naturally in this reduction. In the second step we solve the graviton field equations and identify the holographic boundary data. Finally, we consider the asymptotic form of the black hole space-time and identify the way the information about the mass of the static gravitational configuration is encoded in the holographic data.

## 1 Introduction

In this paper we continue the study started in [10] of the duality between Minkowski space-time and a Conformal Field Theory defined on the boundary of the light-cone. That semiclassical gravity may know something about quantum field theories was first demonstrated by Brown and Henneaux in 1986 [8] who looked at the algebra of gravitational constraints generating the asymptotic symmetries of three-dimensional Anti-de Sitter space-time and found that those constraints form the conformal Virasoro algebra with calculable central charge. This was the first indication in the physical literature that asymptotically Anti-de Sitter space encodes some non-trivial information about conformal symmetry and quantum anomalies in the space one dimension lower. At approximately the same time the mathematicians Fefferman and Graham [17] were interested in the purely mathematical problem of finding possible conformal invariants

and discovered that all such invariants are naturally induced from the ordinary metric invariants near the conformal boundary of the hyperbolic space one dimension higher. The relation between conformal symmetry and the Einstein spaces with negative cosmological constant was established. In their analysis they invented a technical tool of asymptotic expansion, the now famous “Fefferman–Graham expansion”, which later on proved to be very useful in the physical applications.

For almost a decade the two sides (physical and mathematical) of the story were parallel without having close contact with each other. This was until the holographic principle [36], [34] was formulated and a concrete realization of this principle, the AdS/CFT correspondence, was suggested [23], [18], [37]. According to the holographic idea the space-time physics of gravitationally interacting particles should be more economically described in terms of some theory living on the boundary (the so-called “holographic screen”). In the AdS/CFT correspondence the bulk space-time is Anti-de Sitter and the theory on the boundary is a quantum conformal field theory. It is possible to formulate a precise bulk/boundary dictionary translating the (super)gravity phenomena in the bulk to that of CFT on the boundary and vice versa. This led to many interesting developments. Among others, it was understood that there is a deep relation between the geometry of the negative constant curvature space-time and quantum properties of the conformal theories. In particular, the conformal anomalies can be calculated purely geometrically [19] by first expanding the bulk Einstein metric near the conformal boundary and then inserting the expansion back to the gravitational action. The Fefferman–Graham expansion thus made its new appearance and helped to reproduce the Brown–Henneaux central charge in a purely geometrical fashion. The asymptotic diffeomorphisms in Anti-de Sitter space play an important role in generating the conformal symmetry at the boundary [20] and imposing severe constraints on the possible form of the anomalies [28], [29]. Not only conformal anomalies but also the whole structure of the anomalous stress tensor of quantum CFT might be possible to extract from the geometry of hyperbolic space [30], [24]. However, with the exception of the three-dimensional case [30] this is still an open problem. An important element in the holographic description is the way how the hologram should be decoded, i.e. how the bulk gravitational physics is restored from the boundary CFT data. In [12] the necessary holographic data were found to be the metric representing the conformal class on the boundary and the boundary CFT stress tensor. [12] then gives precise prescriptions for how the space-time metric can be reconstructed from these data.

The success of the AdS/CFT duality has motivated the attempts to extend the holographic description to other spaces. It was rather natural to generalize it first to de Sitter space, many elements of this new duality extend straightforwardly from the Anti-de Sitter case while many new subtleties arise [38], [33], [2], [7]. One of the problems is unitarity since typical conformal weights arising in the duality with de Sitter space are complex. Another problem is to formulate the S-matrix description in de Sitter space. Both problems are still open although some suggestions have been made [3], [32].

Minkowski space-time is another important space, a holographic description of which should be understood. A number of ideas and proposals has been circulated in the literature [35]–[21]. In [10] it was suggested to associate the holographic picture with a choice of light-cone in Minkowski space. The part of the space-time which is outside the light-cone is naturally foliated with de Sitter slices while the part which is inside is sliced with the Euclidean Anti-de Sitter spaces. The only boundary of these slices is the boundary of the light-cone itself which is suggested to be the place where the holographic data should be collected. Formulating the holographic dictionary one can make use of the known prescriptions of the AdS/CFT and dS/CFT dualities applying these prescriptions to each separate slice and then summing over all slices. The details of this procedure have been worked out in [10]. In fact, this line of reasoning follows the inspirational paper [17] where the Euclidean hyperbolic space was considered in the context of the cone structure in flat space one dimension higher. The symmetry plays an important role in identifying the way the holographic data should be presented. The Lorentz group of  $(d+2)$ -dimensional Minkowski space becomes the conformal group acting on the  $d$ -sphere lying at (past or future) infinity of the light-cone. Thus the data are expected to have a CFT representation. In this picture it is natural that the propagating near null infinity plane waves are dually described by an infinite set of the conformal operators living on the  $d$ -sphere. Moreover the quantum-mechanical S-matrix can be restored in terms of the correlation functions of operators on two  $d$ -spheres: at infinite past and infinite future on the light-cone.

In the present study we extend the picture suggested in [10] and apply it to the gravitational field itself. More specifically, we want to identify the minimal set of data which has to be specified at the boundary of the light-cone and which is sufficient for the complete reconstruction of the bulk Minkowski space-time. We understand Minkowski space in a wide sense as a Ricci-flat space-time asymptotically approaching the standard flat space structure. It should be noted that there have been earlier attempts in the literature to proceed in a similar direction [5], [4], [11]. The main idea was to integrate the Einstein equations with zero cosmological constant starting with the boundary at spatial infinity and developing the series expansion in the radial direction in a similar fashion Fefferman and Graham have taught us in the case of equations with negative cosmological constant. This program, however, does not work as nicely for asymptotically flat space as it did for asymptotically AdS space. The recurrent relations between terms in the formal series are now differential rather than algebraic as in the AdS case. It is not possible to resolve them and express the coefficients in the series in terms of some boundary data at the spatial infinity. The proposal made in [10] is to start at infinity of the light-cone and integrate the equations from there. One has to develop double expansion in this case: first expansion goes along the constant-curvature slice and the second is in the radial direction enumerating the slices. The relations between coefficients are now algebraic. They can be resolved and the necessary boundary data can be identified. In order to avoid the complexity of non-linear gravitational equations we consider linear perturbations and do the analysis

for the perturbations. This certainly simplifies the problem and provides us with the important information on the non-linear case as well.

This paper is organized as follows. In Section 2 we give some comments on the holographic reconstruction in general emphasizing the role of causality. We use two-dimensional examples to illustrate our point. In Section 3 we review the holographic proposal of [10] in the case of the scalar field. The way the duality works in two-dimensional Rindler space is briefly discussed. We turn to the gravitational case in Section 4 and proceed in two steps. First, we reduce the Minkowski problem to an infinite set of gravitational equations on de Sitter space one dimension lower. These equations are quite remarkable: they describe massless and massive gravitons on de Sitter space. In particular the partially massless graviton appears naturally in this reduction. In the second step we solve each graviton field equation on de Sitter space and identify the boundary data. Decoding the hologram we then have to set the rules and translate the boundary data to the bulk gravitational physics. In Section 5 we make a step in this direction and consider the asymptotic form of the black hole space-time and identify the holographic data which encode the information about the mass of the static gravitational configuration. We conclude with Section 6.

## 2 Holographic reconstruction as a boundary value problem

We start with some general comments on the holographic reconstruction and show that it can be formulated as a (somewhat unusual) boundary value problem. More specifically it is the problem in which the boundary data are entirely specified on time-like or null-like boundary. To make this discussion concrete and simple we take a particular example of massless field in  $(1 + 1)$ -dimensional space-time. Let us first consider the flat space-time with coordinates  $(t, z)$ . The field equation then takes the form

$$-\partial_t^2 \phi + \partial_z^2 \phi = 0. \quad (2.1)$$

Suppose we consider only a part of the space which lies at positive values of the coordinate  $z$ . The standard way to formulate the Cauchy problem in this case would be to specify 1) some initial conditions at  $t = 0$ ,  $\phi(t = 0, z)$  and  $\partial_t \phi(t = 0, z)$ , and 2) the boundary conditions  $\phi(t, z = 0)$  or  $\partial_z \phi(t, z = 0)$ . The necessity to have two pieces of data, one on the “initial surface”  $t = 0$  and another on the boundary  $z = 0$ , follows from a simple causality argument: in order to determine the value of the function  $\phi(t, z)$  at a point  $(t, z)$  we need to have data inside the past-directed light-cone with the tip at the point  $(t, z)$ . For small values of  $t$  the light-cone hits only some part of the “initial surface” and the data on that part is sufficient for determining the value at the point  $(t, z)$ . But for large values of  $t$  the light-cone starts to hit the boundary at  $z = 0$  and the additional data should be specified there. Thus, two pieces of the data come out very naturally in this standard formulation.

In the holographic formulation we would like to have only one piece of data, namely data to be fixed on the time-like boundary  $z = 0$ , and determine the field  $\phi(t, z)$  for all values of  $-\infty < t < +\infty$  and  $z > 0$  from just this data. Simple causality picture considered above in the case of standard formulation helps to visualize the problem in this new formulation. We should again draw a light-cone with the tip at the point  $(t, z)$  but now directed towards the boundary. The boundary data specified on the part of the boundary which lies inside of this light-cone should be sufficient for determining the field  $\phi$  at the point  $(t, z)$ . Another words, in order to reconstruct the field at the point  $(t, z)$  from the data on the boundary ( $z = 0$ ) we should be able to communicate with that point by sending a signal and wait long enough to get the signal back from the point  $(t, z)$ . That's qualitatively how this sort of reconstruction should work. Of course, in order to reconstruct the field for all points  $(t, z)$  we should have at our disposal the all-time boundary data, i.e. for  $-\infty < t < +\infty$ .

As for the question what kind of boundary data should be specified the case of equation (2.1) shows us that we have to specify a pair of functions

$$\phi(t, z = 0) = \psi(t) \quad \text{and} \quad \partial_z \phi(t, z = 0) = \chi(t) \quad (2.2)$$

on the boundary  $z = 0$ . This pair forms the holographic data on the boundary. This is typical situation when the holographic data comes in a pair (function itself and its normal derivative) and we will call this a "holographic pair". The field equation (2.1) subject to the boundary conditions (2.2) can be easily solved and the solution reads

$$\phi(t, z) = \frac{1}{2}[\psi(t + z) + \psi(t - z)] + \frac{1}{2} \int_{t-z}^{t+z} dt' \chi(t'). \quad (2.3)$$

Thus in order to reconstruct the field at a point  $(t, z)$  we have to know boundary value  $\phi(t, z = 0)$  of the field in two moments of time:  $t + z$  and  $t - z$ , and the normal derivative  $\partial_z \phi(t, z = 0)$  on the boundary for all moments of time in-between.

We of course could try to find the solution as an expansion in distance from the boundary. The solution then would take the form of sum of two infinite series

$$\phi(t, z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \psi_t^{(2n)}(t) + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n)!} \chi_t^{(2n)}(t), \quad (2.4)$$

where in each series all coefficients are determined by the boundary function  $\psi(t)$ ,  $\psi_t^{(2n)}(t) = \partial_t^{(2n)} \psi(t)$  and the function  $\chi(t)$ ,  $\chi_t^{(2n+1)}(t) = \partial_t^{(2n+1)} \chi(t)$ . Notice that the causal structure obvious in the complete solution (2.3) is now invisible in the expansion (2.4).

The expansion similar to (2.4) is the standard way to solve the holographic boundary value problem for a field in Anti-de Sitter space. The boundary data are then associated with some CFT data on the boundary. For simplicity let us consider the two-dimensional Anti-de Sitter space with metric

$$ds^2 = \frac{d\rho^2}{4\rho^2} - \frac{1}{\rho} \left(1 - \frac{\rho}{4}\right)^2 dt^2 = g(\rho)[-dt^2 + dz^2], \quad (2.5)$$



with coordinate

$$z(\rho) = \ln \left( \frac{2 + \sqrt{\rho}}{2 - \sqrt{\rho}} \right).$$

The Anti-de Sitter space has time-like boundary located at  $\rho = 0$  or at  $z = 0$  in terms of the coordinate  $z$ . The holographic boundary data thus should be specified there. The metric (2.5) is conformal to two-dimensional flat space-time, actually to a part of it with  $z \geq 0$ . The massless scalar field in two dimensions is conformally invariant so that the solution to the boundary value problem again takes the form (2.3) where  $z$  should be replaced with  $z(\rho)$ . The small  $\rho$ -expansion then has two sorts of terms

$$\phi(t, \rho) = \left[ \psi(t) + \sum_{k=1}^{\infty} F_k(t) \rho^k \right] + \rho^{1/2} \left[ \chi(t) + \sum_{k=1}^{\infty} G_k(t) \rho^k \right], \quad (2.6)$$

where  $\psi(t)$  is the boundary value of the field  $\phi$  and  $\chi(t)$  is the normal derivative of the field at the boundary of Anti-de Sitter. The coefficients  $F_k(t)$  are completely determined by  $\psi(t)$  and its derivatives, and the  $G_k(t)$  are determined by  $\chi(t)$ . Notice again that the causal structure present in solution (2.3) (with  $z = z(\rho)$ ) is lost when we rewrite it in the form of the  $\rho$ -expansion. In the AdS/CFT correspondence the holographic pair  $(\psi(t), \chi(t))$  has the following interpretation:  $\psi(t)$  is associated with the source which couples to a “dual” operator  $\mathcal{O}(t)$  while  $\chi(t)$  should be associated with the quantum expectation value of that operator,  $\chi(t) = \langle \mathcal{O}(t) \rangle$ . The correlation function of the operators at different moments of time can be derived according to the standard AdS/CFT prescription by taking the normal derivatives of the Green’s function

$$D = -\frac{1}{2\pi} \ln \tanh \frac{\sigma}{2}, \quad (2.7)$$

where  $\sigma$  is the geodesic distance between two points on two-dimensional Anti-de Sitter space. The 2-point function on the boundary of  $\text{AdS}_2$  then reads [31]

$$\langle \mathcal{O}(t) \mathcal{O}(t') \rangle \sim \frac{1}{\sinh^2 \left( \frac{t-t'}{2} \right)}. \quad (2.8)$$

The two-dimensional case is of course too simplistic for applying the AdS/CFT dictionary in full since the “boundary” in this case is just a point cross the time. However, this is a good illustration since in higher dimensions the logic in identifying the holographic data and recovering the way of reconstructing the bulk physics from that data is essentially the same. In particular the  $\rho$ -expansion similar to (2.6) is the usual tool for analyzing the reconstruction of supergravity in the bulk from the CFT data on the boundary of Anti-de Sitter. In the case of the gravitational field itself it is actually the only tool available due to extreme non-linearity of the gravitational field equations [19], [12]. We, however, want to emphasize the role of causality in the holographic reconstruction. This role is not obvious when the local series expansion is used.

It is interesting that the boundary should not be necessarily time-like. The holographic boundary problem can be set for a null-like boundary. To illustrate this let us once again exploit our two-dimensional example and consider arbitrary two-dimensional metric which always can be brought to a conformally flat form

$$ds^2 = e^{\sigma(x_+, x_-)} dx_+ dx_- . \quad (2.9)$$

Let us consider the part of the space-time which lies in the corner  $x_- \geq 0$ ,  $x_+ \leq 0$ . The boundary thus consists of two “null surfaces”  $x_+ = 0$  ( $\mathcal{H}_+$ ) and  $x_- = 0$  ( $\mathcal{H}_-$ ). As the boundary data we specify the value of the field function

$$\phi(x_+, x_-)|_{\mathcal{H}_-} = \psi(x_+) \quad \text{and} \quad \phi(x_+, x_-)|_{\mathcal{H}_+} = \chi(x_-) \quad (2.10)$$

on these null-surfaces. Since on the intersection of  $\mathcal{H}_+$  and  $\mathcal{H}_-$  the data should agree we have a constraint,  $\psi(x_+ = 0) = \chi(x_- = 0) = \phi_H$ . The solution of such formulated boundary value problem for the massless scalar field equation then takes a very simple form

$$\phi(x_+, x_-) = \psi(x_+) + \chi(x_-) - \phi_H . \quad (2.11)$$

Thus, in order to reconstruct the field at the point  $(u, v)$  in the bulk we have to know the boundary data at three points on the null boundary: at point  $x_+ = u$  on  $\mathcal{H}_+$ , at point  $x_- = v$  on  $\mathcal{H}_-$  and at the bifurcation point  $H$  ( $x_+ = x_- = 0$ ).

A natural example of the null-surface is the horizon in black hole space-time or de Sitter space. That horizon can play the role of the holographic screen and there might be a dual CFT living on the horizon with bulk/boundary dictionary similar to the one in the case of AdS/CFT correspondence was proposed in [27]. We refer the reader to that paper for further details. Another example when the null-surfaces are natural holographic screens is the Minkowski space-time and the null-screens are the light-cone and null-infinity. This is a possible way of looking at the Minkowski/CFT duality suggested in [10]. We discuss this briefly in the next section.

### 3 Holographic description in Minkowski space

The holographic construction suggested in [10] is associated with a choice of light-cone. The null-surface of a given light-cone  $\mathcal{C}$  naturally splits Minkowski space-time  $\mathcal{M}_{d+2}$  into two regions: the region  $\mathcal{A}$  lying inside the light-cone  $\mathcal{C}$  and the region  $\mathcal{D}$  outside the light-cone. The inside region  $\mathcal{A}$  on the other hand splits into the part which is inside the future light-cone ( $\mathcal{A}_+$ ) and the part which is inside the past light-cone ( $\mathcal{A}_-$ ). Each region admits a natural slicing with constant curvature hypersurfaces. Outside the light-cone it is the slicing with  $(d+1)$ -dimensional de Sitter spaces (which is positive constant curvature space-time with Lorentz signature) while inside the light-cone one may choose the foliation with Euclidean Anti-de Sitter hypersurfaces defined as positive constant curvature space with Euclidean signature. Enumerating the slices

we choose the radial coordinate  $r$  in the region  $\mathcal{D}$  and the time-like coordinate  $t$  in the region  $\mathcal{A}$ . The Minkowski metric then reads

$$\begin{aligned}\mathcal{D}: ds^2 &= dr^2 + r^2(-d\tau^2 + \cosh^2 \tau d\omega^2(\theta)), \\ \mathcal{A}: ds^2 &= -dt^2 + t^2(dy^2 + \sinh^2 y d\omega^2(\theta)),\end{aligned}\tag{3.1}$$

where  $(\tau, \theta)$  and  $(y, \theta)$  are the coordinates on a de Sitter and Anti-de Sitter slice respectively,  $d\omega^2(\theta)$  is a metric on the unit radius  $d$ -sphere with angle coordinates  $\theta$ .

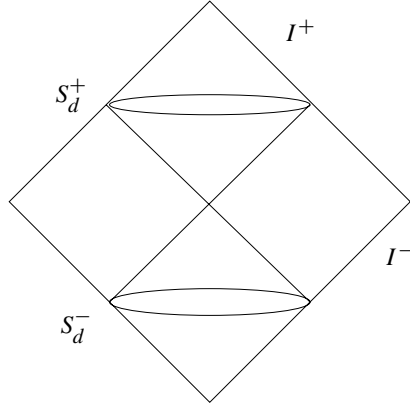


Figure 1. Minkowski space-time with a choice of the light-cone. The boundary sphere  $S_d^+$  ( $S_d^-$ ) lies in the intersection of the light-cone and the future (past) null-infinity.

Each slice in this foliation of Minkowski space-time is a  $(d + 1)$ -dimensional space which has some boundaries. In the Anti-de Sitter case the boundary is the  $d$ -dimensional sphere  $S_d$  lying at infinity of the space while the boundaries of de Sitter space are two spheres  $S_d^+$  and  $S_d^-$  lying respectively in the future and in the past of de Sitter space. The considered foliation has the nice property that all slices have the same boundaries as the light-cone, namely either  $S_d^+$  or  $S_d^-$ . More precisely, all Anti-de Sitter slices covering region  $\mathcal{A}_-$  inside the light-cone have  $S_d^-$  as a boundary while  $S_d^+$  is the only boundary of slices covering region  $\mathcal{A}_+$ . Outside the light-cone, in region  $\mathcal{D}$ , all de Sitter slices have the same boundary  $S_d^+$  in the future and  $S_d^-$  in the past. This property motivated the suggestion made in [10] to associate the holographic information on Minkowski space with these two  $d$ -dimensional spheres. The Lorentz group of  $(d + 2)$ -dimensional Minkowski space acts as a conformal group on the spheres  $S_d^+$  or  $S_d^-$ . So the holographic information is expected to have a conformal field theory description. Some of the details of this holographic description have been demonstrated in [10]. An important element of the description is the specification of the necessary information to be stored on the holographic screens as well as the way how the hologram should be decoded, i.e. the rules of reconstructing the bulk Minkowski physics from the holographic data on the spheres.

This can be illustrated by the example of a massless scalar field<sup>1</sup> described by the field equation  $\nabla^2\phi = 0$ . Let us take for concreteness the region  $\mathcal{D}$  outside the light-cone. Then the field equation reads

$$\left(\partial_r^2 + \frac{(d+1)}{r}\partial_r + \frac{1}{r^2}\nabla_{\text{dS}}^2\right)\phi(r, \tau, \theta) = 0, \quad (3.2)$$

$$\nabla_{\text{dS}}^2 = -\partial_\tau^2 - d \tanh \tau \partial_\tau + \cosh^{-2} \tau \Delta_\theta,$$

where  $\Delta_\theta$  is the Laplace operator on the unit radius  $d$ -sphere. Solving equation (3.2) in the region  $\mathcal{D}$  we expand the field  $\phi(r, \tau, \theta)$  in powers of the radial coordinate  $r$  so that the solution takes the form of the Mellin transform

$$\phi(r, \tau, \theta) = \frac{1}{2\pi i} \int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty} d\lambda r^{-\lambda} \phi_\lambda(\tau, \theta), \quad (3.3)$$

where the functions  $\phi_\lambda(\tau, \theta)$  satisfy the massive wave equation on  $(d+1)$ -dimensional de Sitter space, that is

$$(\nabla_{\text{dS}}^2 - m_\lambda^2)\phi_\lambda(\tau, \theta) = 0, \quad m_\lambda^2 = \lambda(d - \lambda). \quad (3.4)$$

The important question is the range for the spectral parameter  $\lambda$ . A typical configuration in field theory is a plane wave for which the relevant spectral parameter is complex,  $\lambda = \frac{d}{2} + i\alpha$ , with  $\alpha$  changing from minus to plus infinity. This explains the choice of the limits in the integral (3.3). Later on, the parameter  $\lambda$  is identified with the conformal weight of the dual operator. In this case the mass term in (3.4) is real and positive:  $m^2 = \frac{d^2}{4} + \alpha^2$ . Notice that in general the field  $\phi(\tau, r, \theta)$  can be a superposition of propagating modes as well as solitonic configurations. The Coulomb-like potential would be an example of a configuration which requires inclusion in (3.3) of terms with real values of  $\lambda$ . Indeed, in  $d+2$  space-time dimensions the Coulomb-like configuration

$$\phi = \frac{Q}{r^{d-1}(\cosh \tau)^{d-1}} \quad (3.5)$$

corresponds to  $\lambda = d-1$ . We will see that similarly we have to include both complex and real  $\lambda$  in the gravitational case in order to describe both the gravitational waves and black holes. For description of the latter the purely real  $\lambda$  are appropriate. Notice also that in the case  $\lambda = \frac{d}{2}$  the two independent solutions to the radial differential equation (3.2) are  $r^{-d/2}$  and  $r^{-d/2} \ln r$ .

The representation (3.3) is the first step in the holographic reduction: it reduces the quantum field in Minkowski space to a (infinite) set of massive fields living on de Sitter space of one dimension lower. As we have discussed in the beginning of this section, each de Sitter slice has two boundaries,  $S_d^+$  and  $S_d^-$ . So the next step would be to relate each solution of the equation (3.4) to the boundary values of the

---

<sup>1</sup>The consideration is naturally generalized for massive fields and higher spin field equations. The case of Dirac fermions was considered in [22].

functions  $\phi_\lambda(\tau, \theta)$ . Since on the de Sitter space the boundary value problem is in fact the initial value problem, the boundary data should be specified on the surface  $S_d^-$ . The general solution of (3.4) is expressed in terms of  $P$ - and  $Q$ -Legendre functions (see Appendix B)

$$\begin{aligned} \phi_\lambda(\tau, \theta) = (\cosh \tau)^{\frac{1-d}{2}} & \left[ A(\Delta) P_{\frac{(d-1)}{2}-\lambda}^{\frac{\sqrt{(d-1)^2-4\Delta}}{2}} (-i \sinh \tau) \mathcal{O}_\lambda^>(\theta) \right. \\ & \left. + B(\Delta) Q_{\frac{(d-1)}{2}-\lambda}^{\frac{\sqrt{(d-1)^2-4\Delta}}{2}} (-i \sinh \tau) \mathcal{O}_\lambda^<(\theta) \right], \end{aligned} \quad (3.6)$$

where the “constants”  $A(\Delta)$  and  $B(\Delta)$  can be chosen in a way to remove the non-locality in the leading term when expression (3.6) is expanded in powers of  $e^\tau$ .

The expansion in powers of  $e^\tau$  could be used as an alternative way to solve (3.4). The coefficients in front of the terms  $e^{\lambda\tau}$  and  $e^{(d-\lambda)\tau}$  in this expansion are not determined from the equation: these are the initial data to be specified on  $S_d^-$ . In (3.6) these are the functions  $\mathcal{O}_\lambda^>(\theta)$  and  $\mathcal{O}_\lambda^<(\theta)$  which form the “boundary” data. Combining this expansion with the integral representation (3.3) we find that asymptotically near the sphere  $S_d^-$  the solution to the field equation (3.2) in Minkowski space reads

$$\begin{aligned} \phi(r, \tau, \theta) = \frac{1}{2\pi i} \int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty} d\lambda & \left( r^{-\lambda} e^{(d-\lambda)\tau} \left[ \mathcal{O}_\lambda^<(\theta) + \sum_{n=1}^{\infty} \varphi_{(n)\lambda}^<(\theta) e^{2n\tau} \right] \right. \\ & \left. + r^{-\lambda} e^{\lambda\tau} \left[ \mathcal{O}_\lambda^>(\theta) + \sum_{n=1}^{\infty} \varphi_{(n)\lambda}^>(\theta) e^{2n\tau} \right] \right), \end{aligned} \quad (3.7)$$

where the higher order terms in the expansion are uniquely determined by  $\mathcal{O}_\lambda^<(\theta)$  and  $\mathcal{O}_\lambda^>(\theta)$ . The coefficients  $\mathcal{O}_\lambda^<(\theta)$  and  $\mathcal{O}_\lambda^>(\theta)$  are thus the holographic data which are needed to be specified at the sphere  $S_d^-$  for the reconstruction of the scalar field everywhere in the region  $\mathcal{D}$ . These functions can also be associated with the left- and right-moving waves. The sphere  $S_d^-$  lies in the intersection of two null-hypersurfaces: the past null infinity  $\mathcal{I}^-$  and the past light-cone  $\mathcal{C}_-$ . Therefore, instead of two infinite sets of functions on the  $d$ -sphere we may consider just two functions defined on null-space of one dimension higher,

$$\begin{aligned} \mathcal{O}^>(\xi, \theta) &= \frac{1}{2\pi i} \int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty} d\lambda \xi^{-\lambda} \mathcal{O}_\lambda^>(\theta), \\ \mathcal{O}^<(\eta, \theta) &= \frac{1}{2\pi i} \int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty} d\lambda \eta^{-\lambda} \mathcal{O}_\lambda^<(\theta), \end{aligned} \quad (3.8)$$

where  $\xi$  ( $\eta$ ) is the affine parameter along the null infinity  $\mathcal{I}^-$  ( $\mathcal{C}_-$ ).

Similar analysis can be done for the solution near the sphere  $S_d^+$  in the future of the de Sitter slices. A similar infinite set of functions could be specified there. In the quantum mechanical picture the data on the future sphere  $S_d^+$  can be associated

with the quantum out-state while the data specified on  $S_d^-$  form the quantum in-state. In the CFT/Minkowski duality proposed in [10] each coefficient  $\mathcal{O}_\lambda^{>(<)}(\theta)$  appearing in the expansion (3.7) (and in the analogous expansion near  $S_d^+$ ) is associated with a quantum conformal operator of conformal dimension  $\lambda$ . The correlation functions of the in- and out-operators are

$$\begin{aligned} \langle 0|_{\text{out}} \mathcal{O}_{\lambda_1}^{<(>)}(\theta_1)_{\text{in}} \mathcal{O}_{\lambda_2}^{>(<)}(\theta_2)|0\rangle &\sim \delta(\lambda_1 + \lambda_2 - d) \delta^{(d)}(\theta_1, \pi - \theta_2), \\ \langle 0|_{\text{out}} \mathcal{O}_{\lambda_1}^{>(<)}(\theta_1)_{\text{in}} \mathcal{O}_{\lambda_2}^{>(<)}(\theta_2)|0\rangle &\sim \delta(\lambda_1 + \lambda_2 - d) \frac{1}{(1 + \cos \gamma(\theta_1, \theta_2))^{\lambda_2}}, \end{aligned} \quad (3.9)$$

where  $\gamma(\theta, \theta')$  is the geodesic distance between two points on  $d$ -sphere. The S-matrix of the (interacting in the bulk) field can then be reconstructed in terms of the correlation functions between the conformal operators living on  $S_d^+$  and  $S_d^-$ , as was shown in [10].

Things are slightly different inside the light-cone, for instance in the region  $\mathcal{A}_-$  inside the past light-cone. One needs to specify only one set of functions, namely  $\mathcal{O}_\lambda(\theta)$  there. Or, equivalently, only a single function on the past null-infinity (more precisely, on that component of  $\mathcal{I}^-$  which is inside the past light-cone) should be specified. This is quite obvious since  $\mathcal{I}^-$  forms a Cauchy surface in the region  $\mathcal{A}^-$ . In this region the solution to the Cauchy problem can be brought to a nice integral form [10]

$$\begin{aligned} \phi(t, y, \theta) &= \frac{1}{2\pi i} \int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty} d\lambda (-t)^{-\lambda} \int_{S_d^-} d\mu(\theta') G_\lambda(y, \theta, \theta') \mathcal{O}_\lambda(\theta'), \\ G_\lambda(y, \theta, \theta') &= \frac{g(\lambda)}{[\cosh y - \sinh y \cos \gamma(\theta, \theta')]^\lambda}, \end{aligned} \quad (3.10)$$

where  $g(\lambda)$  is some normalization factor,  $d\mu(\theta)$  is the measure on the  $d$ -sphere.  $G_\lambda(y, \theta, \theta')$  has the meaning of boundary-to-bulk propagator on  $\text{AdS}_{d+1}$ .

**Example** ((1+1)-dimensional Minkowski space-time). As an illustration we consider a simple example of two-dimensional Minkowski space with metric

$$ds^2 = -dX_0^2 + dX_1^2. \quad (3.11)$$

The light-cone is defined as  $X_0^2 - X_1^2 = 0$ , the boundary “spheres”  $S^+$  and  $S^-$  are now null-dimensional. In two dimensions the region out-side the light-cone has two components:  $\mathcal{D}_L$  and  $\mathcal{D}_R$  that are analogous to the region  $\mathcal{D}$  in the case of higher-dimensional Minkowski space-time. The region outside the light-cone can be foliated with hyperbolic curves which are the one-dimensional analog of the de Sitter space-time. The foliation is most transparent in new coordinates  $(\tau, r)$  defined as

$$X_0 = r \sinh \tau, \quad X_1 = \pm r \cosh \tau, \quad (3.12)$$

where  $+$  stands for the region  $\mathcal{D}_R$  and  $-$  for the region  $\mathcal{D}_L$ . In these coordinates the metric takes the form

$$ds^2 = -r^2 d\tau^2 + dr^2, \quad (3.13)$$

which can be recognized as two-dimensional Rindler metric. Thus in two dimensions our general holographic construction naturally leads to Rindler space. The solution to the field equation takes the form (3.7) with  $d = 0$  inserted and all  $\varphi_{(n)\lambda}$  vanishing. In the present case there are two sets of operators living on the boundary  $S^+$  or  $S^-$  of the light-cone. These operators,  $\mathcal{O}^>(\omega)$  and  $\mathcal{O}^<(\omega)$ , are associated with left- and right-moving modes there. Notice that there is no angle dependence since  $S^+$  and  $S^-$  are just points. Alternatively, we could consider the operators  $\mathcal{O}^>(\xi)$  and  $\mathcal{O}^<(\eta)$  living on the past null-infinity and the light-cone and defined in (3.8). In this two-dimensional case the reconstruction of the bulk field in terms of the data on these null-surfaces is especially simple and is given by an expression similar to (2.11). The correlation functions of the dual operators can be read off from the structure of the two-function on the Minkowski space when each of the functions is approaching one of the boundaries. Let us restrict our consideration to a simple case of massless field. In this case the Green's function takes the form

$$D(X, X') = \frac{1}{4\pi} \ln s^2(X, X'), \quad (3.14)$$

where the interval  $s^2(X, X')$  between two points in terms of the coordinates  $(\tau, r)$  has the form

$$s^2 = r^2 + r'^2 - 2rr' \cosh(\tau - \tau'). \quad (3.15)$$

The propagator (3.14) has a nice representation in terms of the coordinates  $(\tau, r)$ . For this we first note the representation of the logarithmic function [16]

$$\ln(1 - 2zx + x^2) = -1 - 2 \sum_{n=1}^{\infty} \frac{1}{n} T_n(z) x^n \quad (3.16)$$

in terms of the Tchebycheff polynomials  $T_n(z) = \cos(n \arccos z)$ . Using this representation and replacing the infinite sum in (3.16) with an integral we arrive at another representation for the propagator (3.14)

$$D = \frac{1}{4\pi} \ln r^2 - \frac{1}{4\pi} - \frac{1}{4\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{d\lambda}{\lambda \sin \pi \lambda} \cosh \lambda(\tau - \tau') \left(\frac{r'}{r}\right)^\lambda, \quad (3.17)$$

where we have introduced small  $\epsilon$  in order to avoid the point  $\lambda = 0$  in the integral (3.17). This propagator is in fact a superposition of left- and right-moving modes, which is easily seen after substitution of  $\lambda = i\omega$  into (3.17)

$$D = \frac{1}{2\pi} \ln r - \frac{1}{4\pi} - \frac{1}{8\pi} \int_{-\infty - i\epsilon}^{+\infty - i\epsilon} d\omega K(\omega) \left( e^{i\omega(\tau - \tau')} \left(\frac{r'}{r}\right)^{i\omega} + e^{-i\omega(\tau - \tau')} \left(\frac{r'}{r}\right)^{i\omega} \right), \quad (3.18)$$

where

$$K(\omega) = \frac{2}{\omega} \frac{1}{e^{2\pi\omega} - 1}.$$

From this we find that the correlation function between operators  $\mathcal{O}^>$  and  $\mathcal{O}^<$  reads

$$\langle \mathcal{O}^>(\omega) \mathcal{O}^<(\omega') \rangle = \frac{2}{\omega} \frac{1}{e^{2\pi\omega} - 1} \delta(\omega + \omega'). \quad (3.19)$$

A similar expression is valid for “ $\rangle\rangle$ ” and “ $\langle\langle$ ” correlation functions. It is interesting to note that the function on the right-hand side of (3.19) is exactly the thermal factor for the Hawking radiation. The latter is expected to appear in the Rindler space, the Rindler horizon should be in fact identified with the light-cone we have chosen. It is quite remarkable that the conformal operators living on the boundaries of the horizon (or the light-cone) carry certain information about the Hawking radiation. This observation is another piece of evidence in favor of the so-called “horizon holography” suggested in [27]. We, however, shall not discuss this in the present paper.

## 4 Reconstruction of metric: linear perturbation analysis

The holographic construction being applied to the metric of space-time is somewhat more complicated due to the dual nature of the space-time metric. It defines the dynamics of the gravitational field and also sets the background for other fields. Therefore we should consider a class of metrics which in some sense generalize Minkowski space-time. An appropriate class is that of Ricci-flat metrics. The gravitational equations take the form

$$R_{\mu\nu} = 0. \quad (4.1)$$

These equations are essentially non-linear which makes the analysis more difficult. In addition to the field equations (4.1) one has to specify the boundary conditions, i.e. the asymptotic conditions which should be satisfied by the metric for describing space-time asymptotically approaching the ordinary Minkowski space. The meaning of the words “asymptotically approaching” should be specified as well. A natural (from the perspective of the holographic picture associated with the boundary of the light-cone) condition formulated in [10] is that the space-time should approach the Minkowski structure at least in a vicinity of a  $d$ -sphere lying in the intersection of the null-infinity and the light-cone. A possible way to solve the gravitational equations (4.1) subject to this asymptotic condition is to expand the metric in powers of distance from the sphere (somewhat analogously to what has been done in the case of asymptotically Anti-de Sitter space). In fact, in the present case a double expansion is needed: in powers of  $1/r$  of the inverse radial coordinate enumerating the de Sitter slices close to the sphere  $S_d^-$  and in powers of  $e^\tau$  measuring the distance to the sphere along the de Sitter slice. In general this analysis is very complicated.



We however know that the solution to equation (4.1) asymptotically takes the form [17]

$$\begin{aligned} ds^2 &= dr^2 + r^2 (g_{ij}(\tau, \theta) + O(1/r)) dx^i dx^j, \\ R_{ij}[g] &= d g_{ij}(\tau, \theta), \end{aligned} \quad (4.2)$$

where  $\{x^i\} = \{\tau, \theta\}$ . So asymptotically the Ricci-flat space-time is foliated with constant positive curvature slices generalizing the structure (3.1) of the ordinary Minkowski space-time. This asymptotic structure is further modified by  $1/r$  corrections necessarily present in (4.2). The large  $r$ -expansion however leads to differential relations between the coefficients in the expansion [5], [4], [11]. These relations can not be solved in general. On the other hand, expanding the metric  $g_{ij}(\tau, \theta)$  in powers of  $e^\tau$  we would get the standard asymptotic expansion near the boundary of Einstein space of positive constant curvature. This expansion is algebraic and is similar to the well-known expansion for the hyperbolic Einstein space. In most cases this expansion is an infinite series in  $e^\tau$ . However, when the slice is 3-dimensional the expansion contains a finite number of terms, as was shown in [30]. The boundary  $S_d^-$  of the slice is then two-dimensional and the conformal symmetry is infinite-dimensional. Quite remarkably, this is the case when the asymptotically flat space-time is 4-dimensional. This might be an argument for looking more carefully at the physically interesting 4-dimensional case.

In this paper we take a different route. Instead of dealing with the nonlinearity of the equations (4.1) we look at linear perturbations of (4.1) around Minkowski space with metric (3.1). The holographic analysis then boils down to applying the holographic construction reviewed in the previous section to the equations giving the linear perturbations of (4.1). This is certainly a much simpler problem than solving the non-linear equations, but it also provides us with information on the possible structure of the solution to the non-linear problem and on the necessary holographic data to be specified for the equation (4.1).

Starting the linear perturbation analysis we rewrite the Minkowski metric (3.1) in the form

$$ds^2 = dr^2 + r^2 \gamma_{ij}(x) dx^i dx^j, \quad (4.3)$$

where  $\gamma_{ij}(x)$  is the metric on  $(d+1)$ -dimensional de Sitter space given by

$$\begin{aligned} \gamma_{ij}(x) dx^i dx^j &= -d\tau^2 + e^{A(\tau)} \beta_{ab}(\theta) d\theta^a d\theta^b, \\ A(\tau) &= 2 \ln \cosh \tau, \end{aligned} \quad (4.4)$$

and  $\beta_{ab}(\theta)$  is the metric on the unite radius  $d$ -sphere.

The equation for the linear perturbations  $h_{\mu\nu}(r, x)$  takes the form

$$\nabla^\alpha \nabla_\mu h_{\nu\alpha} + \nabla^\alpha \nabla_\nu h_{\mu\alpha} - \nabla^\alpha \nabla_\alpha h_{\mu\nu} - \nabla_\mu \nabla_\nu \widehat{\text{Tr}} h = 0, \quad (4.5)$$

where  $\widehat{\text{Tr}} h$  and the covariant derivative  $\nabla_\mu$  are defined with respect to the Minkowski metric (4.3). As in the case of scalar field we further proceed in two steps.

#### 4.1 First step: reduction to field equations on de Sitter slice

First we fix the gauge,  $h_{rr} = h_{ri} = 0$ , so that the only non-vanishing components are  $h_{ij}(r, x)$ . In particular, we have that  $\widehat{\text{Tr}} h = r^{-2} \text{Tr} h$ ,  $\text{Tr} h = \gamma^{ij} h_{ij}$ . The equations (4.5) then reduce to a set of equations

$$r^2 \text{Tr} h'' - 2r \text{Tr} h' + 2 \text{Tr} h = 0, \quad (4.6)$$

$$r \partial_r (\nabla^j h_{ij} - \partial_i \text{Tr} h) - 2(\nabla^j h_{ij} - \partial_i \text{Tr} h) = 0, \quad (4.7)$$

$$\begin{aligned} \nabla^k \nabla_i h_{jk} + \nabla^k \nabla_j h_{ik} - \nabla^k \nabla_k h_{ij} - \nabla_i \nabla_j \text{Tr} h - 4h_{ij} \\ - r^2 \partial_r^2 h_{ij} + (3-d)r \partial_r h_{ij} - \gamma_{ij} r^3 \partial_r (r^{-2} \text{Tr} h) = 0. \end{aligned} \quad (4.8)$$

The solution can be taken in the form

$$h_{ij}(r, x) = \sum_{\lambda} r^{2-\lambda} \chi_{ij}^{(\lambda)}(x), \quad (4.9)$$

where the sum (or the integral if appropriate) is taken over all appropriate  $\lambda$ . The analysis is in fact different for  $\lambda = 0$ ,  $\lambda = 1$  and  $\lambda \neq 0, 1$ . Equation (4.6) is satisfied automatically if  $\lambda = 0$  or  $\lambda = 1$  and imposes the condition  $\text{Tr} \chi^{(\lambda)} = 0$  when  $\lambda \neq 0, 1$ . Equation (4.7) is identically satisfied if  $\lambda = 0$  and otherwise imposes the condition

$$\nabla^i \chi_{ij}^{(\lambda)} - \partial_i \text{Tr} \chi^{(\lambda)} = 0, \quad \lambda \neq 0. \quad (4.10)$$

The third equation, (4.8), takes the form

$$\begin{aligned} \nabla^k \nabla_i \chi_{jk}^{(\lambda)} + \nabla^k \nabla_j \chi_{ik}^{(\lambda)} - \nabla^k \nabla_k \chi_{ij}^{(\lambda)} - \nabla_i \nabla_j \text{Tr} \chi^{(\lambda)} \\ + (-2d + m_{\lambda}^2) \chi_{ij}^{(\lambda)} + \lambda \gamma_{ij} \text{Tr} \chi^{(\lambda)} = 0 \end{aligned} \quad (4.11)$$

for any  $\lambda$ , where the mass term  $m_{\lambda}^2 = \lambda(d - \lambda)$  is defined in the same way as for the scalar field (3.4). Notice that in (4.9) all terms can be grouped in pairs,  $r^{2-\lambda} \chi_{ij}^{(\lambda)}$  and  $r^{2-(d-\lambda)} \chi_{ij}^{(d-\lambda)}$ , corresponding to the same mass term  $m^2 = \lambda(d - \lambda)$ . These are two independent solutions to the second order differential equation.

Another case which requires a special treatment is when  $m^2 = \frac{d^2}{4}$  and there is only one  $\lambda = \frac{d}{2}$  which is related to this mass and hence only one radial function  $r^{2-\frac{d}{2}}$ . Since the second order differential equations should have two independent solutions there must be another solution which is not of the form  $r^{2-\lambda}$ . This second solution is  $r^{2-\frac{d}{2}} \ln r$  so that in this case we should search for the solution to the gravitational equations in the form

$$h_{ij} = r^{2-\frac{d}{2}} (\chi_{ij}^{(d/2)}(x) + \varphi_{ij}^{(d/2)}(x) \ln r). \quad (4.12)$$

Inserting this into (4.6), (4.7) and (4.8) we find that the fields  $\chi_{ij}^{(d/2)}$  and  $\varphi_{ij}^{(d/2)}$  decouple from each other in the gravitational equations and are thus independent functions. When  $d \neq 2$  both tensors  $h_{ij}^{(d/2)}(x)$  and  $\varphi_{ij}^{(d/2)}(x)$  are transverse and traceless and

satisfy equation (4.11) with  $m^2 = \frac{d^2}{4}$ . When  $d = 2$  the tensor  $\varphi_{ij}^{(1)}$  is still transverse and traceless while the trace of  $\chi_{ij}^{(d/2)}$  is not restricted.

Putting things together, we find that depending on  $\lambda$  the equations reduce to one of the following form:

$$\begin{aligned} \underline{\lambda = 0}: \quad & \nabla^j \chi_{ij}^{(0)} \text{ and } \text{Tr } \chi^{(0)} \text{ are arbitrary;} \\ & \nabla^k \nabla_i \chi_{jk}^{(0)} + \nabla^k \nabla_j \chi_{ik}^{(0)} - \nabla^k \nabla_k \chi_{ij}^{(0)} - \nabla_i \nabla_j \text{Tr } \chi^{(0)} - 2d \chi_{ij}^{(0)} = 0. \end{aligned} \quad (4.13)$$

$$\underline{\lambda = 1}: \quad \nabla^j \chi_{ij}^{(1)} - \partial_i \text{Tr } \chi^{(1)} = 0, \quad (4.14)$$

$$\begin{aligned} & \nabla^k \nabla_i \chi_{jk}^{(1)} + \nabla^k \nabla_j \chi_{ik}^{(1)} - \nabla^k \nabla_k \chi_{ij}^{(1)} - \nabla_i \nabla_j \text{Tr } \chi^{(1)} \\ & \quad - (d+1) \chi_{ij}^{(1)} + \gamma_{ij} \text{Tr } \chi^{(1)} = 0. \end{aligned} \quad (4.15)$$

$$\underline{\lambda \neq 0, 1}: \quad \nabla^j \chi_{ij}^{(\lambda)} = 0 \quad \text{and} \quad \text{Tr } \chi^{(\lambda)} = 0, \quad (4.16)$$

$$- \nabla^k \nabla_k \chi_{ij}^{(\lambda)} + (2 + m_\lambda^2) \chi_{ij}^{(\lambda)} = 0. \quad (4.17)$$

$\lambda = d/2$ : the solution takes the form (4.12) where depending on value  $d$ , we have one of the following possibilities:

$d \neq 2$ : both  $\chi_{ij}^{(d/2)}$  and  $\varphi_{ij}^{(d/2)}$  satisfy equations (4.16) and (4.17) with  $m^2 = d^2/4$ ;

$d = 2$ : tensor  $\varphi_{ij}^{(d/2)}$  satisfies equations (4.16) and (4.17) with  $m^2 = 1$  while the tensor  $\chi_{ij}^{(d/2)}$  satisfies equations (4.14) and (4.15) and thus should be identified with  $\chi_{ij}^{(1)}$  when  $d = 2$ .

These are equations for  $\chi_{ij}^{(\lambda)}(x)$  considered as some symmetric tensor fields on de Sitter space-time. It is not difficult to recognize that (4.13) is in fact the equation for the massless graviton on de Sitter space. In particular, it is invariant under the usual gauge transformations

$$\chi_{ij}^{(0)} \rightarrow \chi_{ij}^{(0)} + \nabla_i \xi_j + \nabla_j \xi_i. \quad (4.18)$$

The  $\lambda = 0$  perturbations thus describe deformations of Einstein space with positive constant curvature. This is of course consistent with the asymptotic analysis (4.2). The equations (4.17) for linear perturbations characterized by  $\lambda \neq 0, 1$  on the other hand describe the massive graviton on  $(d+1)$ -dimensional de Sitter space. No gauge symmetry is present in this case.

The case  $\lambda = 1$  is a bit trickier. Both equations (4.14) and (4.15) are invariant under gauge transformations

$$\chi_{ij}^{(1)} \rightarrow \chi_{ij}^{(1)} + \nabla_i \nabla_j \xi + \gamma_{ij} \xi \quad (4.19)$$

generated by some scalar function  $\xi(x)$ . This signals that equation (4.15) is some field equation which is already known in the literature. Indeed, it is the equation which describes the spin-two partially massless field<sup>2</sup>. In the context of AdS/CFT and dS/CFT dualities it was considered in [15], [13] and [14]. Note that equation (4.14) arises as a constraint from the field equation (4.15). It is interesting to note that the gauge transformation (4.19) has a natural origin from the Minkowski space perspective. The  $(d + 2)$ -dimensional diffeomorphism preserving the form (4.3) of Minkowski metric takes the following form, as was found in [10],

$$\xi^r = \alpha(x), \quad \xi^i = \frac{1}{r} \gamma^{ij}(x) \partial_j \alpha(x), \quad (4.20)$$

where  $\alpha(x)$  is an arbitrary function. The linear perturbations  $h_{ij}(r, x)$  of the Minkowski metric then change as follows:

$$\delta_\alpha h_{ij} = 2r(\nabla_i \nabla_j \alpha + \gamma_{ij} \alpha). \quad (4.21)$$

So this diffeomorphism acts only on the  $\lambda = 1$  component in  $h_{ij}(r, x)$ , the transformation law of which is identical to (4.19) (after identifying  $\xi = 2\alpha$ ).

Thus, in this first step the gravitational equations on Minkowski space-time reduce to a set of massless and massive graviton field equations on de Sitter space one dimension lower. This is in strict similarity with the scalar field case.

## 4.2 Second step: solving field equations on de Sitter slice

In the next step we want to solve the equations (4.13), (4.15) and (4.17) for all values of  $\lambda$ . One way to do this is to develop an expansion in powers of  $e^\tau$  starting from the boundary  $S_d^-$  on the de Sitter slice. First one takes the linear perturbation of the form  $\chi_{ij}^{(\lambda)}(x) = \sum_\kappa e^{-(\sigma_{ij}-\kappa)\tau} \chi_{ij}^{(\lambda,\kappa)}(\theta)$ , where  $\sigma_{ab} = 2$ ,  $\sigma_{\tau a} = 1$ ,  $\sigma_{\tau\tau} = -2$  and  $\chi_{ij}^{(\lambda,\kappa)}(\theta)$  is a set of functions on  $d$ -sphere, and one then looks for certain values of  $\kappa$  for which the coefficients  $\chi_{ij}^{(\lambda,\kappa)}(\theta)$  are not completely determined by previous terms in the expansion. They form the boundary data to be specified on  $S_d^-$ . In all cases it is found that appropriate values are

$$\kappa = \lambda \quad \text{or} \quad \kappa = d - \lambda.$$

In fact we can do better than just an expansion, we can solve the gravitational field equations exactly. Again the details depend on the value of  $\lambda$ .<sup>3</sup>

**4.2.1  $\lambda = 0$ : Massless graviton in de Sitter space.** The equation for the perturbations in this case is the equation for the massless graviton on de Sitter space which

<sup>2</sup>I thank K. Skenderis for pointing out this to me.

<sup>3</sup>In Subsections 4.2.1 and 4.2.2 we drop the subscript  $\lambda$  in the components  $\chi_{ij}$ . We hope that this should not cause any confusion since the value of  $\lambda$  is explicitly indicated in the heading of each subsection.

has usual gauge freedom (4.18). We choose this freedom to further fix the gauge. Specifically we impose conditions:  $\chi_{\tau\tau} = 0$  and  $\chi_{\tau a} = 0$ ,  $a = 1, 2, \dots, d$ . So the only non-vanishing components are  $\chi_{ab}(\tau, \theta)$ . Then we have that  $\text{Tr } \chi = e^{-A(\tau)} \text{tr } \chi$ ,  $\text{tr } \chi = \beta^{ab} \chi_{ab}$ . The equations (4.13) then take the form

$$e^A \partial_\tau^2 [e^{-A} \text{tr } \chi] - A'^2 \text{tr } \chi + A' \partial_\tau \text{tr } \chi = 0, \quad (4.22)$$

$$\partial_\tau [\nabla^b \chi_{ba} - \partial_a \text{tr } \chi] - A'(\tau) [\nabla^b \chi_{ba} - \partial_a \text{tr } \chi] = 0, \quad (4.23)$$

$$\begin{aligned} e^{-A} [\nabla^c \nabla_a \chi_{bc} + \nabla^c \nabla_b \chi_{ac} - \nabla^c \nabla_c \chi_{ab} - \nabla_a \nabla_b \text{tr } \chi] + \partial_\tau^2 \chi_{ab} \\ + \frac{d-4}{2} A' \partial_\tau \chi_{ab} + \chi_{ab} [A'^2 - 2d] + \frac{1}{2} \beta_{ab} (A' \text{tr } h' - A'^2 \text{tr } h) = 0, \end{aligned} \quad (4.24)$$

where  $\nabla_a$  is the covariant derivative with respect to the metric  $\beta_{ab}(\theta)$  on the sphere  $S_d^-$ . Recall that in (4.22), (4.23), (4.24) we have that

$$e^{A(\tau)} = \cosh^2 \tau.$$

Equations (4.22) and (4.23) are solved immediately, and we find that

$$\begin{aligned} \text{tr } \chi &= B(\theta) \cosh \tau \sinh \tau + D(\theta) \cosh^2 \tau, \\ \nabla^b \chi_{ab} - \nabla_a \text{tr } \chi &= C_a(\theta) e^{A(\tau)}, \end{aligned} \quad (4.25)$$

where the integration constants  $D(\theta)$  and  $B(\theta)$  are some functions on the sphere  $S_d^-$  and  $C_a(\theta)$  is an arbitrary vector field on  $S_d^-$ . As a consequence of (4.25) we have

$$\begin{aligned} \text{tr } \chi' &= A' \text{tr } \chi + B, \\ \text{tr } \chi'' &= (A'' + A'^2) \text{tr } \chi + A' B. \end{aligned} \quad (4.26)$$

Taking the trace of equation (4.24) with respect to the metric  $\beta_{ab}(\theta)$  and using (4.25) and (4.26), we find the relation between the “integration constants”  $C_a(\theta)$  and  $D(\theta)$ :

$$D(\theta) = \frac{1}{d-1} \nabla^a C_a(\theta). \quad (4.27)$$

The metric  $\beta_{ab}$  on the sphere  $S_d^-$  is a maximally symmetric one, for which the curvature tensor reads

$$\begin{aligned} R_{cabd} &= \beta_{cb} \beta_{ad} - \beta_{cd} \beta_{ba}, \\ R_{ab} &= (d-1) \beta_{ab}. \end{aligned} \quad (4.28)$$

Commuting the covariant derivatives with the help of the identity

$$\nabla_c \nabla_a \chi_b^c - \nabla_a \nabla_c \chi_b^c = d \chi_{ab} - \beta_{ab} \text{tr } \chi \quad (4.29)$$

we find that equation (4.24) (after substituting (4.25)) can be written in the form

$$\chi_{ab}'' + \frac{(d-4)}{\cosh \tau} \chi_{ab}' + \frac{(4-2d)}{\cosh^2 \tau} \chi_{ab} - \frac{1}{\cosh^2 \tau} \nabla^2 \chi_{ab} - \frac{1}{\cosh \tau} \mathcal{B}_{ab} + \mathcal{F}_{ab} = 0, \quad (4.30)$$

with

$$\begin{aligned}\mathcal{B}_{ab} &= \beta_{ab}B - \nabla_a \nabla_b B, \\ \mathcal{F}_{ab} &= \nabla_a C_b + \nabla_b C_a + \nabla_a \nabla_b D - 2\beta_{ab}D,\end{aligned}\tag{4.31}$$

and  $D$  defined by (4.27).

As usual the general solution to the inhomogeneous differential equation (4.30) is the sum of a general solution of the homogeneous ( $\mathcal{B}_{ab} = \mathcal{F}_{ab} = 0$ ) equation and a particular solution of the inhomogeneous equation, i.e.

$$\chi_{ab} = \chi_{ab}^{(\text{hom})} + \chi_{ab}^{(\text{inh})}.\tag{4.32}$$

A solution of the inhomogeneous equation can be easily found and it takes the form

$$\chi_{ab}^{(\text{inh})} = \chi_{ab}^{(\mathcal{F})} \cosh^2 \tau + \chi_{ab}^{(\mathcal{B})} \sinh \tau \cosh \tau,\tag{4.33}$$

where

$$\chi_{ab}^{(\mathcal{F})} = \frac{1}{\nabla^2 - 2} \mathcal{F}_{ab}, \quad \chi_{ab}^{(\mathcal{B})} = \frac{1}{d - \nabla^2} \mathcal{B}_{ab}.\tag{4.34}$$

Since  $\text{tr } \mathcal{F} = (\nabla^2 - 2)D$  and  $\text{tr } \mathcal{B} = (d - \nabla^2)B$  we have that

$$\text{tr } \chi^{(\text{inh})} = D \cosh^2 \tau + B \sinh \tau \cosh \tau,$$

and hence (taking into account (4.25)) the homogeneous part in (4.32) should be traceless, that is

$$\text{tr } \chi^{(\text{hom})} = 0.$$

Similarly we can analyze the divergence of (4.33). For this we need to know the commutation relation of the covariant derivative  $\nabla$  and the Laplace type operator  $\nabla^2 = \nabla^a \nabla_a$  as acting on the symmetric tensor  $\chi_{ab}$ . A useful relation is the following:

$$\nabla_b \nabla^2 \chi_a^b - \nabla_a \nabla^2 \text{tr } \chi = \nabla^2 (\nabla_b \chi_a^b - \nabla_a \text{tr } \chi) + (d+1) \nabla_b \chi_a^b + (d-3) \nabla_a \text{tr } \chi.\tag{4.35}$$

Using this relation (and some algebra) we find that (4.34) satisfies

$$\begin{aligned}(1 - \nabla^2)(\nabla^b \chi_{ab}^{(\mathcal{B})} - \partial_a \text{tr } \chi^{(\mathcal{B})}) &= 0, \\ (\nabla^2 + d - 1)(\nabla^b \chi_{ab}^{(\mathcal{F})} - \partial_a \text{tr } \chi^{(\mathcal{F})}) &= (\nabla^2 + d - 1)C_a.\end{aligned}\tag{4.36}$$

Resolving these equations and ignoring the homogeneous part we find that

$$\begin{aligned}(\nabla^b \chi_{ab}^{(\mathcal{B})} - \partial_a \text{tr } \chi^{(\mathcal{B})}) &= 0, \\ (\nabla^b \chi_{ab}^{(\mathcal{F})} - \partial_a \text{tr } \chi^{(\mathcal{F})}) &= C_a(\theta).\end{aligned}\tag{4.37}$$

Thus the homogeneous part in (4.32) should be transverse and traceless,

$$\nabla^b \chi_{ab}^{(\text{hom})} = 0, \quad \text{tr } \chi^{(\text{hom})} = 0.\tag{4.38}$$

Its exact form can be easily found

$$\begin{aligned} \chi_{ab}^{(\text{hom})} = & (\cosh \tau)^{\frac{5-d}{2}} \left[ A_0(\nabla^2) P_{\frac{d-1}{2}}^{\frac{\sqrt{9-2d+d^2-4\nabla^2}}{2}} \left( \sqrt{1 - \cosh^2 \tau} \right) f_{ab}(\theta) \right. \\ & \left. + B_0(\nabla^2) Q_{\frac{d-1}{2}}^{\frac{\sqrt{9-2d+d^2-4\nabla^2}}{2}} \left( \sqrt{1 - \cosh^2 \tau} \right) \psi_{ab}(\theta) \right] \end{aligned} \quad (4.39)$$

where  $P_\nu^\mu(z)$  and  $Q_\nu^\mu(z)$  are Legendre functions and  $f_{ab}(\theta)$  and  $\psi_{ab}(\theta)$  are any transverse-traceless tensors

$$\text{tr } f = \text{tr } \psi = 0, \quad \nabla^a f_{ab} = \nabla^a \psi_{ab} = 0. \quad (4.40)$$

These conditions guarantee that the homogeneous part (4.39) of the solution is transverse and traceless. Although the tracelessness is quite obvious the transverseness should be verified. Indeed, using the identity (4.35) for a traceless tensor we find that

$$\nabla^b \mathcal{P}(\nabla^2) \chi_{ab} = \mathcal{P}(\nabla^2 + d + 1) \nabla^b \chi_{ab} \quad (4.41)$$

is valid for any function  $\mathcal{P}(\nabla^2)$  of the Laplace operator  $\nabla^2$ . Applying this relation to (4.39) we find that the condition  $\nabla^b \chi_{ab}^{(\text{hom})} = 0$  is equivalent to  $\nabla^b f_{ab} = \nabla^b \psi_{ab} = 0$ .

Equation (4.32), where the inhomogeneous part is given by (4.33)–(4.34) and the homogeneous part has the form (4.39), is the general exact solution to the set of gravitational equations (4.22)–(4.24). As it stands, expression (4.39) is highly non-local since it contains a very complicated function of the operator  $\nabla^2$ . However, in the expansion in powers of  $e^\tau$  (when  $\tau \rightarrow -\infty$ ) the few first terms are local.

The “constants”  $A_0(\nabla^2)$  and  $B_0(\nabla^2)$  in (4.39) can be chosen in a way that expansion in powers of  $e^\tau$ , or equivalently in powers of the new variable  $\rho = 4e^{2\tau}$ , takes the form

$$\chi_{ab}^{(\text{hom})} = \frac{1}{\rho} (f_{ab}(\theta) + O(\rho)) + \frac{\rho^{d/2}}{\rho} (\psi_{ab}(\theta) + O(\rho)).$$

Combining this with the analogous expansion for the complete solution (4.32)

$$\chi_{ab} = \frac{1}{\rho} (\chi_{ab}^{(0)}(\theta) + O(\rho)) + \frac{\rho^{d/2}}{\rho} (\chi_{ab}^{(d)}(\theta) + O(\rho)),$$

where  $\chi_{ab}^{(0)}$  has the meaning of deformation of the metric on the sphere  $S_d^-$  and  $\chi_{ab}^{(d)}$  is related to the stress tensor of the dual CFT living on the sphere  $S_d^-$ , we find that

$$\begin{aligned} \chi_{ab}^{(0)} &= f_{ab}(\theta) + \frac{1}{\nabla^2 - 2} \mathcal{F}_{ab} + \frac{1}{\nabla^2 - d} \mathcal{B}_{ab}, \\ \chi_{ab}^{(d)} &= \psi_{ab}(\theta). \end{aligned} \quad (4.42)$$

Thus the so far undetermined “integration constants”  $C_a(\theta)$  and  $B(\theta)$  in (4.25) can be related to the trace and divergence of the deformation  $\chi_{ab}^{(0)}$

$$\begin{aligned} C_a &= \nabla^b \chi_{ab}^{(0)} - \partial_a \text{tr} \chi^{(0)}, \\ B &= \frac{1}{(d-1)} (\nabla^a \nabla^b \chi_{ab}^{(0)} - \nabla^2 \text{tr} \chi^{(0)}) - \text{tr} \chi^{(0)}. \end{aligned} \quad (4.43)$$

The first equation in (4.42) can be viewed as a way to represent an arbitrary symmetric tensor  $\chi_{ab}^{(0)}$  in terms of its trace, divergence and the transverse-traceless part.

The  $d = 2$  case is special. In this case there takes place the following

**Statement.** *If  $\chi_{ab}$  is any tensor on two-dimensional manifold such that  $\text{tr} \chi = 0$  and  $\nabla^b \chi_{ab} = 0$  then*

$$\nabla^2 \chi_{ab} = R \chi_{ab},$$

where  $R$  is the scalar curvature of the manifold.

This statement can be verified by brut-force calculation. For the sphere we have  $R = 2$  so that any transverse-traceless tensor in two dimensions is an eigenfunction of the Laplace operator  $\nabla^2$  with the eigenvalue 2. This means that we can make a substitution  $\nabla^2 = 2$  everywhere in (4.39). The Legendre functions then become trigonometric functions and the homogeneous part of the solution reads

$$\chi_{ab}^{(\text{hom})} = \tilde{f}_{ab} \cosh^2 \tau + \tilde{\psi}_{ab} \cosh \tau \sinh \tau. \quad (4.44)$$

Combining this with the inhomogeneous part (4.33) we find that the total solution (4.32) in two dimensions being expressed in terms of the variable  $\rho$  has only few terms. This is of course consistent with the more general result obtained in [30] that in  $d + 1 = 3$  the solution to Einstein’s equations with nonzero cosmological constant has  $\rho$ -expansion which terminates after the first three terms. Here we have proven this for the perturbations. The proof however is rather non-trivial since it was not quite clear from the expression (4.39) how the complicated Legendre functions may reduce to just few exponential terms even after we put  $d = 2$  in (4.39). The above Statement was crucial for the demonstration of the consistency.

Summarizing this subsection, the arbitrary symmetric tensor  $\chi_{ab}^{(0)}(\theta)$  describing the deformation of the metric structure on the  $d$ -sphere and the transverse-traceless tensor  $\psi_{ab}(\theta)$  related to the stress tensor of the dual CFT are the holographic data to be specified on the  $d$ -sphere  $S_d^-$ , which completely determine the  $(d + 2)$ -dimensional Ricci-flat metric in the sector  $\lambda = 0$ .

In the holographic pair  $(\chi_{ab}^{(0)}, \psi_{ab})$  the function  $\chi_{ab}^{(0)}$  represents a source which on the boundary  $S_d^-$  couples to a dual operator represented by  $\psi_{ab}(\theta)$ . The coupling then is

$$\int_{S_d^-} \chi_{ab}^{(0)} \psi^{ab}. \quad (4.45)$$



The gauge invariance (4.18) is usual coordinate invariance on the  $d$ -sphere

$$\delta_\xi \chi_{ab}^{(0)} = \nabla_a \xi_b + \nabla_b \xi_a,$$

where  $\xi$  is a vector on  $S_d^-$  under which (4.45) is supposed to be invariant. This imposes the constraint  $\nabla^a \psi_{ab} = 0$  on the dual operator, which also motivates its interpretation as a stress tensor. This condition is what we also get by solving the massless graviton field equation (see (4.40)).

**4.2.2  $\lambda = 1$ : Partially massless graviton in de Sitter space.** The equations for perturbations in this case are collected in Appendix B. As was discussed above these equations describe a partially massless graviton field in de Sitter space of dimension  $d+1$ . This equation has gauge symmetry (4.19). In order to fix the gauge-independent degrees of freedom we may want to impose some gauge conditions. A possible condition to impose is

$$\text{Tr } \chi = -\chi_{\tau\tau} + e^A \text{tr } \chi = 0.$$

It is the gauge suggested in [15]. Another possible way to impose gauge fixing the constraint is to demand that

$$\chi_{\tau\tau} = 0. \quad (4.46)$$

Looking at the transformation for the component  $\chi_{\tau\tau}$ ,

$$\delta_\xi \chi_{\tau\tau} = \partial_\tau^2 \xi - \xi, \quad (4.47)$$

we find that condition (4.46) restricts the gauge parameter  $\xi(\theta, \tau)$  to take the form

$$\xi = \xi_0(\theta)e^{-\tau} + \xi_2(\theta)e^\tau. \quad (4.48)$$

In this subsection we prefer to use condition (4.46) and we will see that the field equations are considerably simplified in this gauge. As we can see from (4.48), the condition (4.46) does not fix the components  $\xi^{(0)}$  and  $\xi^{(2)}$  of the gauge parameter so that there still remains some fiducial gauge invariance. In fact this invariance is important and plays the role similar to the asymptotic conformal symmetry in the case  $\lambda = 0$ . On the  $(\tau a)$ - and  $(ab)$ -components of the perturbation the gauge transformation with parameter taking the form (4.48) acts as follows:

$$\delta_\xi \chi_{\tau a} = \frac{1}{\cosh \tau} (\partial_a \xi_0 - \partial_a \xi_2), \quad (4.49)$$

$$\delta_\xi \chi_{ab} = e^{-\tau} \left[ \nabla_a \nabla_b \xi_0 + \frac{1}{2} \beta_{ab} (\xi_0 + \xi_2) \right] + e^\tau \left[ \nabla_a \nabla_b \xi_2 + \frac{1}{2} \beta_{ab} (\xi_2 + \xi_0) \right].$$

In the gauge (4.46) the equations of Appendix C are simplified and can be solved explicitly. Substituting equation (C.1) into (C.3) and recalling that  $e^A = \cosh^2 \tau$  we find that  $\text{tr } \chi = \beta^{ab} \chi_{ab}$  satisfies the simple differential equation

$$\text{tr } \chi'' - \text{tr } \chi = 0. \quad (4.50)$$

The general solution is

$$\text{tr } \chi = \alpha(\theta) \cosh \tau + \gamma(\theta) \sinh \tau, \quad (4.51)$$

where  $\alpha(\theta)$  and  $\gamma(\theta)$  are some integration constants. Substituting this back into equations (C.1) and (C.2) we get that

$$\nabla^a \chi_{a\tau} = \frac{\gamma(\theta)}{\cosh \tau} \quad (4.52)$$

and

$$(\nabla^b \chi_{ba} - \partial_a \text{tr } \chi) = \cosh^2 \tau (\partial_\tau \chi_{\tau a} + d \frac{\sinh \tau}{\cosh \tau} \chi_{\tau a}). \quad (4.53)$$

Taking one more divergence of (4.53) we get

$$\nabla^a \nabla^b \chi_{ab} - \nabla^2 \text{tr } \chi = (d-1) \gamma(\theta) \sinh \tau. \quad (4.54)$$

Equations (4.51) and (4.54) tell us that in the expansion of the perturbation  $\chi_{ab}$  in powers of  $e^\tau$  all terms, except the first three terms, are traceless and partially conserved. The partial conservation is very important (see [15]) in the theory of partially a massless graviton field and for its relation to a conformal field theory on the boundary. As was discussed in [15], the partial conservation is directly related to the gauge symmetry<sup>4</sup> generated by (4.48). We discuss this point later in the paper. Here we just note that the functions  $\alpha(\theta)$  and  $\gamma(\theta)$  transform as

$$\begin{aligned} \delta_\xi \alpha(\theta) &= (\nabla^2 + d)(\xi_0 + \xi_2), \\ \delta_\xi \gamma(\theta) &= \nabla^2(\xi_2 - \xi_0). \end{aligned} \quad (4.55)$$

These functions are the only variables which transform non-trivially under the gauge transformation (4.19).

The next equation to be solved is (C.4). Substituting there the gauge condition (4.46), equation (C.2) and explicit expression for  $A(\tau)$  we find that this equation takes the simpler form

$$\chi''_{\tau a} + d \frac{\sinh \tau}{\cosh \tau} \chi'_{\tau a} + [d-1 - \frac{(\nabla^2 - 1)}{\cosh^2 \tau}] \chi_{\tau a} + \frac{\partial_a \gamma}{\cosh^3 \tau} = 0. \quad (4.56)$$

This equation can be solved explicitly and the general solution is a sum of a particular solution of the inhomogeneous equation and the general solution of the homogeneous equation, that is

$$\chi_{\tau a} = \chi_{\tau a}^{(\text{inh})} + \chi_{\tau a}^{(\text{hom})}, \quad (4.57)$$

where we find that

$$\chi_{\tau a}^{(\text{inh})} = \frac{1}{\cosh \tau} \frac{1}{(\nabla^2 - d + 1)} \partial_a \gamma \quad (4.58)$$

---

<sup>4</sup>In [15] only the part of transformations which is due to  $\xi_0$  was considered.

and

$$\begin{aligned} \chi_{\tau a}^{(\text{hom})} = (\cosh \tau)^{\frac{1-d}{2}} & \left[ A(\nabla^2) P_{\frac{d-3}{2}}^{\frac{\sqrt{5-2d+d^2-4\nabla^2}}{2}} (-i \sinh \tau) J_a(\theta) \right. \\ & \left. + B(\nabla^2) Q_{\frac{d-3}{2}}^{\frac{\sqrt{5-2d+d^2-4\nabla^2}}{2}} (-i \sinh \tau) I_a(\theta) \right]. \end{aligned} \quad (4.59)$$

This solution should satisfy equation (4.52) and thus we get some conditions on the so far arbitrary “constants”  $J_a(\theta)$  and  $I_a(\theta)$ . Using identity (A.5) we show that

$$\nabla^a \chi_{a\tau}^{(\text{inh})} = \frac{\gamma(\theta)}{\cosh \tau}. \quad (4.60)$$

So the homogeneous part of the solution should be covariantly conserved,  $\nabla^a \chi_{a\tau}^{(\text{hom})} = 0$ . Using identity (A) we find that the latter condition imposes the constraints

$$\nabla^a J_a(\theta) = \nabla^a I_a(\theta) = 0, \quad (4.61)$$

i.e.  $J_a$  and  $I_a$  are arbitrary covariantly conserved vectors on  $S_d^-$ . Choosing  $A(\nabla^2)$  and  $B(\nabla^2)$  appropriately we find that (4.57) has an asymptotic expansion

$$\chi_{\tau a} = 2e^\tau (\chi_{\tau a}^{(0)} + O(e^\tau)) + e^{\tau(d-1)} (I_a(\theta) + O(e^\tau)), \quad (4.62)$$

where the rest terms in the expansion are completely determined by these two terms, and we have that

$$\chi_{\tau a}^{(0)} = J_a + \frac{1}{\nabla^2 - d + 1} \partial_a \gamma, \quad \nabla^a \chi_{\tau a}^{(0)} = \gamma(\theta). \quad (4.63)$$

This is a way to present a vector in terms of its divergence ( $\gamma$ ) divergence-free part ( $J_a$ ). Thus two vectors, the arbitrary vector  $\chi_{\tau a}^{(0)}$  and the divergence-free vector  $I_a$ , form the first holographic pair at the level  $\lambda = 1$ .

The only equation left is equation (C.5) on the components  $\chi_{ab}$  of the perturbation. After all substitutions made this equation reads

$$\begin{aligned} \chi_{ab}'' + (d-4) \frac{\sinh \tau}{\cosh \tau} \chi_{ab}' + \left[ 3 - d - \left( \frac{4 - 2d + \nabla^2}{\cosh^2 \tau} \right) \right] \chi_{ab} \\ + 2 \frac{\sinh \tau}{\cosh \tau} [\nabla_a \chi_{\tau a} + \nabla_b \chi_{\tau b}] + \frac{1}{\cosh \tau} [\nabla_a \nabla_b \alpha - \beta_{ab} \alpha] \\ + \frac{\sinh \tau}{\cosh^2 \tau} [\nabla_a \nabla_b \gamma - 2\beta_{ab} \gamma] = 0. \end{aligned} \quad (4.64)$$

This is again an inhomogeneous equation, the terms staying in the second and third line of (4.64) play the role of the source for the differential operator staying in the first line. The solution is again of the familiar form

$$\chi_{ab} = \chi_{ab}^{(\text{inh})} + \chi_{ab}^{(\text{hom})}, \quad (4.65)$$

where the homogeneous part takes the form

$$\begin{aligned} \chi_{ab}^{(\text{hom})} = & (\cosh \tau)^{\frac{5-d}{2}} \left[ A_1(\nabla^2) P_{\frac{d-3}{2}}^{\frac{\sqrt{9-2d+d^2-4\nabla^2}}{2}} (-i \sinh \tau) k_{ab}(\theta) \right. \\ & \left. + B_1(\nabla^2) Q_{\frac{d-3}{2}}^{\frac{\sqrt{9-2d+d^2-4\nabla^2}}{2}} (-i \sinh \tau) p_{ab}(\theta) \right]. \end{aligned} \quad (4.66)$$

Several terms contribute to the inhomogeneous part

$$\chi_{ab}^{(\text{inh})} = \chi_{ab}^{(\alpha)} + \chi_{ab}^{(\gamma)} + \chi_{ab}^{(J)} + \chi_{ab}^{(I)}, \quad (4.67)$$

where

$$\chi_{ab}^{(\alpha)} = \frac{1}{\nabla^2 - d} (\nabla_a \nabla_b - \beta_{ab}) \alpha(\theta) \cosh \tau, \quad (4.68)$$

$$\begin{aligned} \chi_{ab}^{(\gamma)} = & \frac{2}{\nabla^2 - 2d + 4} \left( \frac{1}{2} \nabla_a \nabla_b - \beta_{ab} \right. \\ & \left. + \nabla_a \frac{1}{\nabla^2 - d + 1} \nabla_b + \nabla_b \frac{1}{\nabla^2 - d + 1} \nabla_a \right) \gamma(\theta) \sinh \tau, \end{aligned} \quad (4.69)$$

$$\chi_{ab}^{(J)} = \mathcal{F}^{(J)}(\nabla^2, \tau) (\nabla_a J_b + \nabla_b J_a), \quad (4.70)$$

$$\chi_{ab}^{(I)} = \mathcal{F}^{(I)}(\nabla^2, \tau) (\nabla_a I_b + \nabla_b I_a).$$

The function  $\mathcal{F}^{(J)}(\nabla^2, \tau)$  ( $\mathcal{F}^{(I)}(\nabla^2, \tau)$ ) is a solution to the differential equation

$$\mathcal{F}'' + (d-4) \frac{\sinh \tau}{\cosh \tau} \mathcal{F}' + \left( 3 - d - \frac{(4-2d+\nabla^2)}{\cosh^2 \tau} \right) \mathcal{F} + 2 \frac{\sinh \tau}{\cosh \tau} \Phi = 0 \quad (4.71)$$

with

$$\Phi^{(J)}(\nabla^2, \tau) = (\cosh \tau)^{\frac{1-d}{2}} A(\nabla^2 - d - 1) P_{\frac{d-3}{2}}^{\frac{\sqrt{9+2d+d^2-4\nabla^2}}{2}} (-i \sinh \tau)$$

and

$$\Phi^{(I)}(\nabla^2, \tau) = (\cosh \tau)^{\frac{1-d}{2}} B(\nabla^2 - d - 1) Q_{\frac{d-3}{2}}^{\frac{\sqrt{9+2d+d^2-4\nabla^2}}{2}} (-i \sinh \tau).$$

Equation (A.5) was used in deriving  $\Phi^{(I)}$  and  $\Phi^{(J)}$  from (4.59). We do not have a closed-form expression for the functions  $\mathcal{F}^{(J)}$  and  $\mathcal{F}^{(I)}$ , but the expansion is readily available:

$$\begin{aligned} \mathcal{F}^{(J)}(\nabla^2, \tau) &= e^\tau \left[ \frac{1}{4-d} + O(e^\tau) \right], \\ \mathcal{F}^{(I)}(\nabla^2, \tau) &= e^{\tau(d-1)} \left[ \frac{1}{d} + O(e^\tau) \right], \end{aligned} \quad (4.72)$$

where we keep only the leading terms. The dependence on  $\nabla^2$  appears in the subleading terms. Two cases are special:  $d = 2$  and  $d = 4$ . The expansion should then be

modified:

$$\begin{aligned}\mathcal{F}^{(J,I)}(\nabla^2, \tau) &= e^\tau \tau^2 [1 + O(e^\tau)] \quad (d=2), \\ \mathcal{F}^{(J)}(\nabla^2, \tau) &= e^\tau \tau [1 + O(e^\tau)], \quad (d=4).\end{aligned}\tag{4.73}$$

In terms of  $\rho$  it involves a logarithm,  $\rho \ln \rho$  and  $\rho^{1/2} \ln \rho$  respectively. As (4.72) and (4.73) indicate,  $\chi_{ab}^{(J)}$  and  $\chi_{ab}^{(I)}$  contribute so that  $J_a$  and  $I_a$  show up in the subleading terms of the total solution (4.65).

Using identities from Appendix A we can now show that

$$\begin{aligned}\nabla^a \nabla^b \chi_{ab}^{(\alpha)} - \nabla^2 \text{tr} \chi^{(\alpha)} &= 0, \\ \nabla^a \nabla^b \chi_{ab}^{(J)} = \nabla^a \nabla^b \chi_{ab}^{(I)} &= 0, \\ \nabla^a \nabla^b \chi_{ab}^{(\gamma)} - (\nabla^2 + d - 1) \text{tr} \chi^{(\gamma)} &= 0.\end{aligned}\tag{4.74}$$

It indicates that the non-conservation in equation (4.54) is entirely due to the term  $\chi_{ab}^{(\gamma)}$ . Similarly for the trace we have that

$$\text{tr} \chi^{(\alpha)} = \alpha(\theta) \cosh \tau, \quad \text{tr} \chi^{(\gamma)} = \gamma(\theta) \sinh \tau, \quad \text{tr} \chi^{(J)} = \text{tr} \chi^{(I)} = 0.\tag{4.75}$$

Combining (4.74) and (4.75) with (4.51) and (4.54) we conclude that the homogeneous part of the solution should be traceless and partially conserved, so  $\text{tr} \chi^{(\text{hom})} = \nabla^a \nabla^b \chi_{ab}^{(\text{hom})} = 0$ . This gives conditions

$$\text{tr} k = \text{tr} p = 0 \quad \text{and} \quad \nabla^a \nabla^b k_{ab} = \nabla^a \nabla^b p_{ab} = 0\tag{4.76}$$

for the integration constants  $k_{ab}(\theta)$  and  $p_{ab}(\theta)$  in (4.66).

Choosing appropriately  $A_1(\nabla^2)$  and  $B_1(\nabla^2)$  we find that (4.65) has an expansion

$$\chi_{ab} = \frac{1}{2} e^{-\tau} (\chi_{ab}^{(0)}(\theta) + O(e^\tau)) + e^{(d-3)\tau} (p_{ab}(\theta) + O(e^\tau)),\tag{4.77}$$

where the rest terms are determined by these two terms and by the functions  $J_a$  and  $I_a$  which appear in the terms starting with  $e^\tau$  and  $e^{(d-1)\tau}$ , respectively. The leading term  $\chi_{ab}^{(0)}$  in (4.77) has the meaning of boundary value of the perturbation; we have that

$$\begin{aligned}\chi_{ab}^{(0)} &= \frac{1}{\nabla^2 - d} (\nabla_a \nabla_b - \beta_{ab}) \alpha + \frac{2}{\nabla^2 - 2d + 4} \left( \frac{1}{2} \nabla_a \nabla_b - \beta_{ab} \right. \\ &\quad \left. + \nabla_a \frac{1}{\nabla^2 - d + 1} \nabla_b + \nabla_b \frac{1}{\nabla^2 - d + 1} \nabla_a \right) \gamma(\theta) + k_{ab}.\end{aligned}\tag{4.78}$$

Thus the functions  $\alpha$  and  $\gamma$  can be related to the trace and the partial non-conservation of the tensor  $\chi_{ab}^{(0)}$

$$\begin{aligned}\alpha(\theta) &= 2 \text{tr} \chi^{(0)} + \frac{1}{d-1} [\nabla^a \nabla^b \chi_{ab}^{(0)} - \nabla^2 \text{tr} \chi^{(0)}], \\ \gamma(\theta) &= -\frac{1}{d-1} [\nabla^a \nabla^b \chi_{ab}^{(0)} - \nabla^2 \text{tr} \chi^{(0)}],\end{aligned}\tag{4.79}$$

so that (4.78) is just a way to represent any symmetric tensor in terms of its trace, partial non-conservation and a traceless and partially conserved part.

We have not yet used equation (4.53). This equation imposes certain relations between the so far independent functions  $\chi_{ab}^{(0)}$ ,  $p_{ab}$ ,  $\chi_{\tau a}^{(0)}$  and  $I_a$  appearing in the expansions (4.77) and (4.62). Substituting these expansions into equation (4.53) and comparing the terms of the same order of  $e^\tau$  on both sides we find the relations

$$\nabla^b \chi_{ab}^{(0)} - \partial_a \text{tr} \chi^{(0)} = (1 - d) \chi_{\tau a}^{(0)} \quad (4.80)$$

and

$$\nabla^b p_{ab} = -\frac{1}{4} I_a. \quad (4.81)$$

Together with (4.57) equations (4.65)–(4.69) give us the exact and complete solution to the partially massless graviton field equations on de Sitter space. We are now in the position to determine the holographic data on the boundary ( $S_d^-$ ) of de Sitter space. In addition to the pair  $(\chi_{\tau a}^{(0)}, I_a)$  the two functions  $(\chi_{ab}^{(0)}, p_{ab})$  form another holographic pair at the level  $\lambda = 1$ . This data is subject to constraints (4.61), (4.76) and (4.80) and (4.81). Notice that the partial conservation is a consequence of relations (4.80) and (4.81) and of condition (4.61). This completes the holographic data at this level.

In the dS/CFT duality in each pair  $(\chi_{\tau a}^{(0)}, I_a)$  and  $(\chi_{ab}^{(0)}, p_{ab})$  the first function should be considered as a source which couples on the boundary ( $S_d^-$ ) to a quantum operator associated with the second function of the pair. The couplings thus take the form

$$\int_{S_d^-} \chi_{\tau a}^{(0)} I^a \quad \text{and} \quad \int_{S_d^-} \chi_{ab}^{(0)} p^{ab}.$$

The gauge invariance (4.49) for the source

$$\begin{aligned} \delta_\xi \chi_{\tau a}^{(0)} &= 2\partial_a (\xi_2 - \xi_0), \\ \delta_\xi \chi_{ab}^{(0)} &= (\nabla_a \nabla_b \xi_0 + \frac{1}{2} \beta_{ab} \xi_0) + \frac{1}{2} \beta_{ab} \xi_2 \end{aligned}$$

then implies that the dual operators should satisfy certain constraints: the vector  $I_a(\theta)$  should have vanishing divergence and the tensor  $p_{ab}(\theta)$  should be traceless and partially conserved. This is exactly what we see from our solution (see (4.61) and (4.76)). Concluding this subsection we want to stress that the description dual to the partially massless graviton in de Sitter space does not just contain a tensor operator  $p_{ab}(\theta)$  which is traceless and partially conserved, as it was suggested in [15]. It should contain also a divergence-free vector operator  $I_a(\theta)$  related to the operator  $p_{ab}(\theta)$  according to (4.81).

**4.2.3  $\lambda \neq 0, 1$ : Massive graviton in de Sitter space.** The process of solving the field equations in this case goes pretty much in a similar fashion as before. One of the

field equations (or rather constraints) is that the perturbation should be traceless (see second equation in (4.16)). This equation allows to express the component  $\chi_{\tau\tau}^\lambda$  of the perturbation in terms of the trace of components  $\chi_{ab}^\lambda$  in the following way:

$$\chi_{\tau\tau}^{(\lambda)} = e^{-A(\tau)} \operatorname{tr} \chi^{(\lambda)}. \quad (4.82)$$

The first equation in (4.16) then gives a pair of equations (where (4.82) has been taken into account)

$$\nabla^a \chi_{\tau a}^{(\lambda)} = \partial_\tau \operatorname{tr} \chi^{(\lambda)} + \frac{(d-1)}{2} A' \operatorname{tr} \chi^{(\lambda)} \quad (4.83)$$

and

$$\nabla^b \chi_{ba}^\lambda = e^A \left( \frac{d}{2} A' \chi_{\tau a}^\lambda + \partial_\tau \chi_{\tau a}^\lambda \right). \quad (4.84)$$

The field equations (4.17) are collected in Appendix D. Notice that the  $(\tau\tau)$  component of (4.17) is an equation for the trace  $\operatorname{tr} \chi^\lambda$ . With the help of (4.83) this equation takes the form

$$\partial_\tau^2 \operatorname{tr} \chi^\lambda + \frac{d}{\cosh \tau} \partial_\tau \operatorname{tr} \chi^\lambda + [\lambda(d-\lambda) - \frac{\nabla^2}{\cosh^2 \tau}] \operatorname{tr} \chi^\lambda = 0 \quad (4.85)$$

and in fact is identical to the scalar field equation (3.4) on de Sitter space considered in Section 2. The solution is similar to (3.6)

$$\begin{aligned} \operatorname{tr} \chi^\lambda = (\cosh \tau)^{\frac{1-d}{2}} & \left[ A_\lambda^{(0)}(\nabla^2) P_{\frac{(d-1)}{2}-\lambda}^{\frac{\sqrt{(d-1)^2-4\nabla^2}}{2}} (-i \sinh \tau) f^\lambda(\theta) \right. \\ & \left. + B_\lambda^{(0)}(\nabla^2) Q_{\frac{(d-1)}{2}-\lambda}^{\frac{\sqrt{(d-1)^2-4\nabla^2}}{2}} (-i \sinh \tau) g^\lambda(\theta) \right], \end{aligned} \quad (4.86)$$

where  $A_\lambda^{(0)}(\nabla^2)$  and  $B_\lambda^{(0)}(\nabla^2)$  are chosen in a way such that the asymptotic behavior of (4.86) has the form

$$\operatorname{tr} \chi^\lambda = e^{\tau(d-\lambda)} (f^\lambda(\theta) + O(e^\tau)) + e^{\tau\lambda} (g^\lambda(\theta) + O(e^\tau)). \quad (4.87)$$

Using (4.82) the asymptotic expansion for the  $(\tau\tau)$  components of the perturbation is

$$\chi_{\tau\tau}^\lambda = 4e^{2\tau} (e^{\tau(d-\lambda)} (f^\lambda(\theta) + O(e^\tau)) + e^{\tau\lambda} (g^\lambda(\theta) + O(e^\tau))). \quad (4.88)$$

The  $(\tau a)$  component (D.2) of equation (4.17) takes a similar form of the inhomogeneous differential equation

$$\partial_\tau^2 \chi_{\tau a}^\lambda + \frac{d}{\cosh \tau} \partial_\tau \chi_{\tau a}^\lambda + \left[ \lambda(d-\lambda) - \frac{(\nabla^2-1)}{\cosh^2 \tau} \right] \chi_{\tau a}^\lambda + 2 \frac{\sinh \tau}{\cosh^3 \tau} \partial_a \operatorname{tr} \chi^\lambda = 0. \quad (4.89)$$

The solution is the sum of two terms

$$\chi_{\tau a} = \chi_{\tau a}^{(\text{hom})} + \chi_{\tau a}^{(\text{inh})}, \quad (4.90)$$

where

$$\begin{aligned} \chi_{\tau a}^{(\text{hom})} = & (\cosh \tau)^{\frac{1-d}{2}} \left[ A_{\lambda}^{(1)} (\nabla^2) P_{\frac{(d-1)}{2}-\lambda}^{\frac{\sqrt{d^2-2d+5-4\nabla^2}}{2}} (-i \sinh \tau) J_a^{(\lambda)}(\theta) \right. \\ & \left. + B_{\lambda}^{(1)} (\nabla^2) Q_{\frac{(d-1)}{2}-\lambda}^{\frac{\sqrt{d^2-2d+5-4\nabla^2}}{2}} (-i \sinh \tau) I_a^{(\lambda)}(\theta) \right], \end{aligned} \quad (4.91)$$

and  $I_a^{(\lambda)}(\theta)$  and  $J_a^{(\lambda)}(\theta)$  are (at this point) arbitrary vectors on the  $d$ -sphere. The term  $\chi_{\tau a}^{(\text{inh})}$  in (4.90) is due to the inhomogeneity in (4.89) caused by the term depending on  $\text{tr } \chi^\lambda$ . Although a closed-form expression for  $\chi_{\tau a}^{(\text{inh})}$  may be possible to find what we really need is an expansion of the solution in powers of  $e^\tau$ ,

$$\begin{aligned} \chi_{\tau a}^{(\text{inh})}(\tau, \theta) = & 4e^{2\tau} \left( e^{\tau(d-\lambda)} \left( \frac{\partial_a f^\lambda(\theta)}{2+d-2\lambda} + O(e^\tau) \right) \right. \\ & \left. + e^{\tau\lambda} \left( \frac{\partial_a g^\lambda(\theta)}{2\lambda+2-d} + O(e^\tau) \right) \right). \end{aligned} \quad (4.92)$$

These terms are subleading with respect to those coming from the expansion of (4.91) so that for the total solution (4.90) we have that

$$\chi_{\tau a}^\lambda = e^{\tau(d-\lambda)} (I_a^{(\lambda)}(\theta) + O(e^\tau)) + e^{\tau\lambda} (J_a^{(\lambda)}(\theta) + O(e^\tau)) . \quad (4.93)$$

The equation (D.3)

$$\begin{aligned} \chi_{ab}'' + \frac{(d-4)}{\cosh \tau} \chi_{ab}' + \left[ \lambda(d-\lambda) + 4 - 2d - \left( \frac{4-2d+\nabla^2}{\cosh^2 \tau} \right) \right] \chi_{ab} \\ + 2 \frac{\sinh \tau}{\cosh \tau} [\nabla_a \chi_{\tau a} + \nabla_b \chi_{\tau a}] - 2 \frac{\sinh^2 \tau}{\cosh^2 \tau} \beta_{ab} \text{tr } \chi = 0 \end{aligned} \quad (4.94)$$

has a solution in the form

$$\chi_{ab}^\lambda = \chi_{ab}^{(\text{hom})} + \chi_{ab}^{(\text{inh})}, \quad (4.95)$$

where the homogeneous part is

$$\begin{aligned} \chi_{ab}^{(\text{hom})} = & (\cosh \tau)^{\frac{5-d}{2}} \left[ A_{\lambda}^{(2)} (\nabla^2) P_{\frac{d-1}{2}-\lambda}^{\frac{\sqrt{9-2d+d^2-4\nabla^2}}{2}} (-i \sinh \tau) f_{ab}^\lambda(\theta) \right. \\ & \left. + B_{\lambda}^{(2)} (\nabla^2) Q_{\frac{d-1}{2}-\lambda}^{\frac{\sqrt{9-2d+d^2-4\nabla^2}}{2}} (-i \sinh \tau) \psi_{ab}^\lambda(\theta) \right]. \end{aligned} \quad (4.96)$$

Its expansion (when  $\tau \rightarrow -\infty$ ) is

$$\chi_{ab}^{(\text{hom})} = e^{-2\tau} (e^{\tau(d-\lambda)} (\psi_{ab}^{(\lambda)}(\theta) + O(e^\tau)) + e^{\tau\lambda} (f_{ab}^\lambda(\theta) + O(e^\tau))). \quad (4.97)$$

The expansion of the inhomogeneous part in (4.95) can be obtained by inserting the expansions for  $\text{tr } \chi^\lambda$  (4.87) and  $\chi_{\tau a}^\lambda$  (4.93) into equation (4.94) and taking the leading



part in the equation. We then obtain

$$\chi_{ab}^{(\text{inh})} = \frac{1}{2\lambda - d + 2} H_{ab}^\lambda e^{\lambda\tau} + \frac{1}{d + 2 - 2\lambda} G_{ab}^\lambda e^{(d-\lambda)\tau}, \quad (4.98)$$

where we keep only the leading terms and define

$$\begin{aligned} H_{ab}^\lambda &= \nabla_a I_b^\lambda + \nabla_b I_a^\lambda + \beta_{ab} g^\lambda, \\ G_{ab}^\lambda &= \nabla_a J_b^\lambda + \nabla_b J_a^\lambda + \beta_{ab} f^\lambda. \end{aligned}$$

Obviously (4.98) is subleading with respect to (4.97). Thus asymptotically the leading order the  $(ab)$  components of the perturbation behave as

$$\chi_{ab}^\lambda = e^{-2\tau} (e^{\tau(d-\lambda)} \psi_{ab}^{(\lambda)}(\theta) + e^{\tau\lambda} f_{ab}^\lambda(\theta)). \quad (4.99)$$

At this point we see that the complete solution to the set of equations (4.82), (D.1)–(D.3) is characterized by the set of tensors  $f_{ab}^\lambda$ ,  $\psi_{ab}^\lambda$ ,  $J_a^\lambda$ ,  $I_a^\lambda$ ,  $g^\lambda$  and  $f^\lambda$  defined on the sphere  $S_d^-$ . Now we have to take into account the equations (4.83) and (4.84). This will impose certain relations between these functions. Indeed, substituting the expansions (4.87), (4.93) and (4.97) into (4.83) and (4.84) and looking at the leading order we get the relations

$$\nabla^a J_a^\lambda = (\lambda - d + 1) g^\lambda \quad \text{and} \quad \nabla^a I_a^\lambda = (1 - \lambda) f^\lambda \quad (4.100)$$

and

$$\nabla^b f_{ab}^\lambda = \frac{(\lambda - d)}{4} J_a^\lambda \quad \text{and} \quad \nabla^b \psi_{ab}^\lambda = -\frac{\lambda}{4} I_a^\lambda. \quad (4.101)$$

More relations come from the consistency condition that the trace of (4.95) should coincide with (4.86). Comparing the leading terms in the expansions (4.87) and (4.99) we find that  $\psi_{ab}^\lambda$  and  $f_{ab}^\lambda$  should be traceless,

$$\text{tr } \psi^\lambda = \text{tr } f^\lambda = 0. \quad (4.102)$$

This also means that the whole homogeneous part (4.96) should be traceless so that the non-vanishing trace  $\text{tr } \chi^\lambda$  (4.86) is entirely due to the inhomogeneous part in (4.95). To the leading order it can be checked directly (using the relations (4.100) and expansions (4.98) and (4.87)).

Several remarks are in order. First, it should be noted that all holographic data satisfying the relations (4.100) and (4.101) can be grouped in pairs: the one which corresponds to  $\lambda$  and another one which corresponds to  $(d - \lambda)$ . Interestingly it can be extended to include the cases  $\lambda = 0$  and  $\lambda = 1$  which are not of the massive graviton case. Then, the pair to the massless graviton  $\lambda = 0$  is the massive graviton with  $\lambda = d$  and the pair to the partially massless graviton  $\lambda = 1$  is the massive graviton with  $\lambda = d - 1$ . Of course, these pairs are just two independent solutions to the second order radial differential equations discussed in Section 4.1. Second, a somewhat degenerate case is  $\lambda = \frac{d}{2}$ . Then the two independent asymptotic solutions considered in this subsection should be  $e^{\tau \frac{d}{2}}$  and  $e^{\tau \frac{d}{2} \tau}$ . We do not consider in detail

this case. Finally, we note that the asymptotic form (4.99) multiplied by the radial function  $r^{2-\lambda}$  with complex  $\lambda = \frac{d}{2} + i\alpha$  is exactly what one would have expected for the representation of plane gravitational waves and is similar to the representation (3.7) for the plane waves in the case of the scalar field.

### 4.3 Summary

Let us summarize our rather long analysis. The metric perturbation over Minkowski space has been represented in the form

$$h_{ij}(r, \tau, \theta) = \sum_{(\lambda)} r^{2-\lambda} \chi_{ij}^{\lambda}(\tau, \theta) + r^{2-\frac{d}{2}} (\chi_{ij}^{(d/2)}(x) + \varphi_{ij}^{(d/2)}(x) \ln r),$$

where the sum over  $\lambda$  may contain also an integral as it happens when  $\lambda = \frac{d}{2} + i\alpha$  with continuous  $\alpha$ . Also, the case of  $\lambda = \frac{d}{2}$  is special and involves a logarithm in the representation for the perturbation. This term is written in the above expression explicitly.

$\lambda = 0$ : the components  $\chi_{\tau\tau}$  and  $\chi_{\tau a}$  vanish by the gauge conditions; the asymptotic behavior for  $(ab)$  components is

$$\chi_{ab} = e^{-2\tau} [(\chi_{ab}^{(0)} + \dots) + (e^{(d-1)\tau} \psi_{ab} + \dots)],$$

where  $\chi_{ab}^{(0)}(\theta)$  is arbitrary and has the meaning of deformation of the metric structure on the  $d$ -sphere;  $\psi_{ab}(\theta)$  should satisfy conditions

$$\text{tr } \psi = 0 \quad \text{and} \quad \nabla^b \psi_{ab} = 0.$$

$\lambda = 1$ :  $\chi_{\tau\tau} = 0$  by gauge fixing; the asymptotic behavior of the non-vanishing components is given by

$$\begin{aligned} \chi_{\tau a} &= 2e^{\tau} (\chi_{\tau a}^{(0)}(\theta) + \dots) + e^{\tau(d-1)} (I_a(\theta) + \dots), \\ \chi_{ab} &= \frac{1}{2} e^{-\tau} [(\chi_{ab}^{(0)}(\theta) + \dots) + e^{(d-3)\tau} (p_{ab}(\theta) + \dots)], \end{aligned}$$

where the following conditions should be satisfied:

$$\nabla^a I_a = 0 \quad \text{and} \quad \text{tr } p = 0, \quad \nabla^a \nabla^b p_{ab} = 0$$

as well as the relations

$$\nabla^b \chi_{ab}^{(0)} - \partial_a \text{tr } \chi^{(0)} = (1-d) \chi_{\tau a}^{(0)} \quad \text{and} \quad \nabla^b p_{ab} = -\frac{1}{4} I_a.$$

$\lambda \neq 0, 1$ : all components are non-vanishing, the asymptotic behavior is given by

$$\begin{aligned}\chi_{\tau\tau}^\lambda &= 4e^{2\tau} [e^{\tau(d-\lambda)} (f^\lambda(\theta) + \dots) + e^{\tau\lambda} (g^\lambda(\theta) + \dots)], \\ \chi_{\tau a}^\lambda &= e^{\tau(d-\lambda)} [(I_a^{(\lambda)}(\theta) + \dots) + e^{\tau\lambda} (J_a^\lambda(\theta) + \dots)], \\ \chi_{ab}^\lambda &= e^{-2\tau} [e^{\tau(d-\lambda)} (\psi_{ab}^{(\lambda)}(\theta) + \dots) + e^{\tau\lambda} (f_{ab}^\lambda(\theta) + \dots)],\end{aligned}$$

with constraints and relations

$$\begin{aligned}\text{tr } \psi^\lambda &= \text{tr } f^\lambda = 0, \\ \nabla^a J_a^\lambda &= (\lambda - d + 1)g^\lambda \quad \text{and} \quad \nabla^a I_a^\lambda = (1 - \lambda)f^\lambda, \\ \nabla^b f_{ab}^\lambda &= \frac{(\lambda - d)}{4} J_a^\lambda \quad \text{and} \quad \nabla^b \psi_{ab}^\lambda = -\frac{\lambda}{4} I_a^\lambda.\end{aligned}$$

$\lambda = \frac{d}{2}$ : If  $d \neq 2$  both functions  $\chi_{ij}^{(d/2)}$  and  $\psi_{ij}^{(d/2)}$  have the same expansion as in the case  $\lambda \neq 0, 1$  considered above; in the case  $d = 2$  the function  $\chi_{ij}^{(d/2)}$  should be identified with the  $\lambda = 1$  perturbation.

The coefficients in the above expansions (subject to the above mentioned constraints) form the holographic data on the sphere  $S_d^-$ , which should be sufficient for the complete reconstruction of the  $(d+2)$ -dimensional Ricci-flat metric. The uncovered holographic data should have an interpretation in terms of the conformal field theory living on the sphere  $S_d^-$  as well as from the point of view of  $(d+2)$ -dimensional gravitational physics. As for the CFT interpretation the tensors  $f_{ab}^\lambda(\theta)$  and  $\psi_{ab}^\lambda(\theta)$  are naturally regarded as an infinite set of the stress tensors corresponding to the infinite set of the conformal operators on the  $d$ -sphere that represent the matter degrees of freedom. That the stress tensors are not conserved indicates that the dual conformal theory couples to a set of sources (represented by operators  $J_a^\lambda$  and  $I_a^\lambda$ ). In this case the conservation is replaced by the Ward identity (see [12] for some discussion of this). On the other hand, the operators  $J_a^\lambda$  and  $I_a^\lambda$  are not conserved as well due to coupling to the operators  $g^\lambda$  and  $f^\lambda$ . It would be interesting to make more precise the relation between these operators and the infinite set of operators ( $\mathcal{O}_\lambda^>$  and  $\mathcal{O}_\lambda^<$ ) representing the matter fields. For that one would have to analyze the coupled gravity-matter system.

It seems natural to suggest that the holographic data should encode information about the mass and rotation of the asymptotically flat gravitational configuration. In particular, we expect that  $I_a^\lambda$  and  $J_a^\lambda$  should carry information about the angular momentum. Also, the data should contain information about the energy flow coming through the null-infinity. The latter seems to be encoded in the data corresponding to  $\lambda = \frac{d}{2} + i\alpha$ . The details however need to be further understood. In the next section we solve a somewhat simpler problem and analyze where in the holographic data on  $S_d^-$  the information about the mass of the static gravitational configuration is stored.

## 5 Asymptotic form of the black hole metric

The holographic data which we revealed in the previous section should encounter for all relevant information about the gravitational physics in asymptotically flat space-time. In particular it should encode the energy balance between the mass of the gravitational configuration and the energy flow coming through the null-infinity. Thus, we expect that the Bondi mass can be appropriately redefined in terms of the described holographic data. This is a problem for future investigation. Here we solve a simpler problem of encoding the information about the mass of a static configuration. We want to see which particular term in the  $\lambda$ -representation of the asymptotic metric contains information about the mass. We start with the known metric of a (non-rotating) black hole and then bring it to the form which is more appropriate for our asymptotic analysis. The standard form of the metric is

$$ds^2 = -g(\rho)dt^2 + g^{-1}(\rho)d\rho^2 + \rho^2 d\omega_{S_d}^2, \quad (5.1)$$

where the metric function  $g(\rho)$  depends on the space-time dimension. When the space-time dimension is  $d + 2 = 4$ , the metric (5.1) is the Schwarzschild solution

$$g(\rho) = 1 - \frac{2m}{\rho}. \quad (5.2)$$

In higher dimensions the metric is known as Myers–Perry metric [25]

$$g(\rho) = 1 - \frac{2m}{\rho^{d-1}}. \quad (5.3)$$

The parameter  $m$  is the mass in the case  $d = 2$  and is related to the mass when  $d > 2$ . The metric (5.1) should be brought to the form

$$ds^2 = dr^2 + r^2(-F(r, \tau)d\tau^2 + R(r, \tau)d\omega_{S_d}^2) \quad (5.4)$$

in terms of new coordinates  $(r, \tau)$ . For  $r \rightarrow \infty$  the function  $F(r, \tau)$  should approach 1 while the function  $R(r, \tau)$  equals  $\cosh^2 \tau$  in this limit so that the standard form of the flat space-time metric is restored. In this limit the relation between the coordinates  $(\rho, t)$  and  $(r, \tau)$  is  $\rho = r \cosh \tau, t = r \sinh \tau$ .

That the metric (5.1) is non-flat manifests in modifying these relations by subleading terms. The subleading terms appear in the  $r$ -expansion of the functions  $R(r, \tau)$  and  $F(r, \tau)$ , respectively. It is rather straightforward though quite tedious to obtain this expansion. Below we present the result.

$d = 2$ :

$$\begin{aligned} F(r, \tau) &= 1 + \frac{4m}{\cosh^3 \tau} \frac{\ln r}{r} + \frac{2}{r} f(\tau), \\ R(r, \tau) &= \cosh^2 \tau - \frac{2m}{\cosh \tau} \frac{\ln r}{r} + \frac{2}{r} a(\tau) \cosh \tau. \end{aligned} \quad (5.5)$$

$d > 2$ :

$$\begin{aligned} F(r, \tau) &= \left(1 + \frac{f(\tau)}{r}\right)^2 - \left(\frac{d}{d-2}\right) \frac{2m}{\cosh^{d+1} \tau} \frac{1}{r^{d-1}}, \\ R(r, \tau) &= \left(\cosh \tau + \frac{a(\tau)}{r}\right)^2 + \frac{2m}{(d-2) \cosh^{d-1} \tau} \frac{1}{r^{d-1}}. \end{aligned} \quad (5.6)$$

We skip the subleading terms in (5.5) and (5.6). In both cases the function  $f(\tau)$  is arbitrary, and  $a(\tau)$  is defined as

$$a(\tau) = \int f(\tau) \sinh \tau \, d\tau.$$

As we have already seen, the  $d = 2$  case (Minkowski space-time has physical dimension 4) is in many respects special. Equation (5.5) indicates another peculiarity of  $d = 2$ . Quite surprisingly, the  $r$ -expansion of the Schwarzschild metric starts with a logarithmic term,  $r^{-1} \ln r$ . Also we see that in both (5.5) and (5.6) the mass makes its first appearance at the level  $\lambda = d - 1$ . To make it conform to our notation let us rewrite the expansions (5.5) and (5.6) in the form

$$\begin{aligned} r^2(\varphi_{ij}^{(1)} r^{-1} \ln r + \chi_{ij}^{(1)} r^{-1}), \quad d = 2, \\ r^2(\chi_{ij}^{(1)} r^{-1} + \chi_{ij}^{(d-1)} r^{-(d-1)}), \quad d > 2, \end{aligned} \quad (5.7)$$

where

$$\begin{aligned} \varphi_{\tau\tau}^{(1)} &= \frac{4m}{\cosh^3 \tau}, \quad \varphi_{ab}^{(1)} = -\frac{2m}{\cosh \tau} \beta_{ab}(\theta), \\ \chi_{\tau\tau}^{(d-1)} &= -\left(\frac{d}{d-2}\right) \frac{2m}{\cosh^{d+1} \tau}, \quad \chi_{ab}^{(d-1)} = \frac{2m}{(d-2) \cosh^{d-1} \tau} \beta_{ab}(\theta). \end{aligned} \quad (5.8)$$

In both cases the components of  $\chi_{ij}^{(1)}$  are

$$\chi_{\tau\tau}^{(1)} = 2f(\tau), \quad \chi_{ab}^{(1)} = 2a(\tau) \cosh \tau \beta_{ab}(\theta). \quad (5.9)$$

Comparing the expressions (5.8) with our analysis we see that the components (5.8) are what we called the inhomogeneous part (4.98) of the perturbation while the homogeneous part vanishes identically in (5.8). Notice also that the  $\lambda = d - 1$  solution to the radial differential equations corresponds to the mass term  $m^2 = d - 1$ . The second independent solution is the one with  $\lambda = 1$ . In the case  $d = 2$  the two independent solutions which correspond to the mass term  $m^2 = 1$  are  $r \ln r$  and  $r$ . We see that in the expansion (5.6) there appear both independent solutions corresponding to  $m^2 = d - 1$ : the  $\lambda = d - 1$  term (the  $r \ln r$  term in the  $d = 2$  case) contains information about the mass of the gravitational configuration while the  $\lambda = 1$  term contains an arbitrary function of  $\tau$ . Freedom in the choice of this function is apparently a manifestation of the gauge symmetry (4.19) appearing exactly at the level  $\lambda = 1$ . Comparing (5.8) with our analysis in Section 4.2.3 we can single out the primary element in the holographic

data which contains the information about the mass. We find that it is the function  $g^{(\lambda=1)}(\theta)$  which is proportional to the mass  $m$  and brings the dependence on  $m$  into all metric components. Generically,  $g^{(\lambda)}(\theta)$  (defined in (4.87)) is a source for the vector  $J_a^\lambda$  via equation (4.100). But in the case  $\lambda = 1$  the coefficient in front of the source vanishes, and  $J_a^\lambda$  is divergence-free. On the other hand, the current  $J_a^\lambda$  plays the role of the source for  $f_{ab}^\lambda$  (4.101) and since  $J_a^\lambda$  is conserved it means that  $f_{ab}^\lambda$  is partially conserved. (It is also traceless by (4.102).) This latter property singles out the value  $\lambda = d - 1$ . Of course, all  $J_a^\lambda$  and  $f_{ab}^\lambda$  identically vanish in the solution (5.5) and (5.6). But they would be non-trivial in the general case of dynamical situation when there is flow of energy coming through the past null-infinity. It is certainly interesting to analyze how information about the flow is encoded in the holographic data.

## 6 Conclusion

Minkowski space-time can be reconstructed from some data specified on the boundary of light-cone. This works in a fashion similar to the known holographic reconstruction of asymptotically Anti-de Sitter space from the boundary data. In the latter case the holographic pair consists of metric representing the conformal class on the boundary and the stress tensor of the boundary theory. In the present case, since infinitely many AdS and dS slices end at the boundary of the light-cone, we should expect that infinitely many stress tensors need to be specified there. Also, since the data should represent the  $(d + 2)$ -dimensional physics which does not confine to the boundary only, these stress tensors are not expected to be conserved. Rather they should satisfy the Ward identities as required by the coordinate invariance. Indeed, what we have found is the chains of holographic operators that can be represented as follows:

$$f_{ab}^\lambda \xrightarrow{\nabla} J_a^\lambda \xrightarrow{\nabla} g^\lambda$$

and

$$\psi_{ab}^\lambda \xrightarrow{\nabla} I_a^\lambda \xrightarrow{\nabla} f^\lambda.$$

The label  $\lambda$  parameterizes the infinite family of such operators. Another expectation is that in the dual theory the central charge (coming naively from the radius of each (A)dS slice) should be a continuous function perhaps parameterized by  $\lambda$  so that each stress tensor in the infinite family should have its own central charge. However, we do not see this in our analysis since all tensors  $f_{ab}^\lambda$  and  $\psi_{ab}^\lambda$  are traceless. Of course, these tensors represent only the linearized part of the stress tensors in the full non-linear problem. But that they are traceless in the linear order may indicate that the central charge is well-defined in the dual theory independently of  $\lambda$ . This should be further investigated. From the gravitational perspective the above sets of operators represent the bulk gravitational dynamics and should describe the mass, the angular momentum and the energy flow. The latter can be due to gravitational waves and is

likely to be represented by the holographic operators with complex  $\lambda$ . We finish by remarking that our construction may be a nice starting point for the quantization of the asymptotically flat gravitational field since the set of the holographic data is obviously even-dimensional and seems to be well-suited for the introduction of the symplectic structure.

**Acknowledgments.** The work on this project took an old-fashionedly long interval of time during which the author's daughter Alexandra was born and the author has managed to move to a new University. It is a pleasure to acknowledge the relaxing and stimulating atmosphere (very much helpful for a technically involved and long-lasting project) of the group of Slava Mukhanov at LMU, München, where this project was started. The author thanks Jan de Boer for the fruitful collaboration on paper [10]. The discussions with M. Anderson, J. Barbon, O. Biquard, R. Graham, K. Krasnov, R. Mann, R. Myers, I. Sachs, K. Schleich, K. Skenderis and L. Smolin were very useful. The preliminary results were reported at the 73rd Meeting between Physicists and Mathematicians on "(A)dS/CFT correspondence" (Strasbourg), September 2003. Special thanks to the organizers for organizing this wonderful meeting and for their kind persistence which played crucial role in the completion of this work. At various stages this project was supported by DFG grants SPP 1096 and Schu 1250/3-1.

## Appendix

### A Some useful identities in $dS_d$ and $S_d$

In this appendix we collect some useful commutation relations of the covariant derivative and the Laplace type operator  $\nabla^2 = \nabla^a \nabla_a$  and a metric  $\beta_{ab}$  on the  $d$ -dimensional de Sitter space. These identities are valid in any signature. In the case of Euclidean signature the space is a  $d$ -dimensional sphere. Both spaces are maximally symmetric so that the Riemann curvature can be expressed in terms of a metric  $\beta_{ab}$ :

$$\begin{aligned} R_{cabd} &= \beta_{cb}\beta_{ad} - \beta_{cd}\beta_{ba}, \\ R_{ab} &= (d-1)\beta_{ab}. \end{aligned} \tag{A.1}$$

The commutation of covariant derivatives on such spaces is significantly simplified. The useful relations are:

$$\nabla_a \nabla^2 \phi = \nabla^2 \nabla_a \phi - (d-1) \nabla_a \phi \tag{A.2}$$

for a scalar field  $\phi$ ,

$$\nabla_a \nabla^2 A_b = (\nabla^2 - d + 1) \nabla_a A_b + 2\beta_{ab} \nabla^c A_c - 2\nabla_b A_a \tag{A.3}$$

for a vector field  $A_a$ . Contracting the indices in (A.3) we find that

$$\nabla^a \nabla^2 A_a = (\nabla^2 + d - 1) \nabla^a A_a. \tag{A.4}$$

Taking the symmetrization of equation (A.3) and assuming that  $\nabla^a A_a = 0$  we get another useful identity for vector:

$$\nabla_a \nabla^2 A_b + \nabla_b \nabla^2 A_a = (\nabla^2 - d - 1)(\nabla_a A_b + \nabla_b A_a). \quad (\text{A.5})$$

For a symmetric tensor  $h_{ab}$  we get the identity

$$\nabla_b \nabla^2 h_a^b = (\nabla^2 + d + 1)(\nabla_b h_a^b) - 2\nabla_a \text{tr } h. \quad (\text{A.6})$$

## B Legendre functions: differential equation and asymptotic behavior

A solution to the differential equation

$$y''(\tau) + \frac{1-2a}{\cosh \tau} y'(\tau) - \left(b - \frac{c}{\cosh^2 \tau}\right) y(\tau) = 0, \quad (\text{B.1})$$

where  $a, b, c$  are some constants, is a combination of  $P$ - and  $Q$ -Legendre functions:

$$y(\tau) = (\cosh(\tau))^a \left( C_1 P_{-\frac{1}{2} + \sqrt{b+(a-\frac{1}{2})^2}}^{\sqrt{a^2+c}}(-i \sinh \tau) + C_2 Q_{-\frac{1}{2} + \sqrt{b+(a-\frac{1}{2})^2}}^{\sqrt{a^2+c}}(-i \sinh \tau) \right), \quad (\text{B.2})$$

where  $C_1$  and  $C_2$  are arbitrary integration constants.

For large  $\xi$  the Legendre functions asymptotically behave as follows [16]:

$$\begin{aligned} P_\nu^\mu(\xi) &= \frac{(2\xi)^\nu}{\sqrt{\pi}} \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\nu - \mu + 1)} \left( 1 + O\left(\frac{\ln \xi}{\xi}\right) \right), \quad \text{Re } \nu > -\frac{1}{2} \\ Q_\nu^\mu(\xi) &= \frac{\sqrt{\pi}}{(2\xi)^{\nu+1}} \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\nu + \frac{3}{2})} \left( 1 + O\left(\frac{\ln \xi}{\xi}\right) \right) \end{aligned} \quad (\text{B.3})$$

## C $\lambda = 1$ equations on the de Sitter space $dS_{d+1}$

Keeping all components of  $\chi_{ij}$ , i.e.  $\chi_{\tau\tau}$ ,  $\chi_{\tau a}$  and  $\chi_{ab}$ , the equation (4.14) splits into two equations

$$\frac{d}{2} A' \chi_{\tau\tau} = e^{-A} (\nabla^a \chi_{a\tau} + \frac{1}{2} A' \text{tr } \chi - \partial_\tau \text{tr } \chi), \quad (\text{C.1})$$

$$\partial_a \chi_{\tau\tau} - \partial_\tau \chi_{\tau a} - \frac{d}{2} A' \chi_{\tau a} = e^{-A} (\partial_a \text{tr } \chi - \nabla^b \chi_{ba}), \quad (\text{C.2})$$

where  $\text{tr } \chi = \beta^{ab} \chi_{ab}$  and  $\nabla^a$  is with respect to a metric  $\beta_{ab}$  on the  $d$ -sphere.



The other group of equations comes from (4.15). For  $(\tau\tau)$ ,  $(\tau a)$  and  $(ab)$  components of (4.15) we have that

$$\begin{aligned} & \frac{d}{2} A' \partial_\tau \chi_{\tau\tau} - \frac{d}{2} (A'^2 - 2) \chi_{\tau\tau} \\ & + e^{-A} \left( \text{tr } \chi - \frac{A'^2}{2} \text{tr } \chi + e^A \partial_\tau^2 (e^{-A} \text{tr } \chi) - \nabla^c \nabla_c \chi_{\tau\tau} + 2A' \nabla^b \chi_{\tau b} \right) = 0, \end{aligned} \quad (\text{C.3})$$

$$\begin{aligned} & \partial_\tau^2 \chi_{\tau a} + A' \frac{(d-2)}{2} \partial_\tau \chi_{\tau a} + \left[ \frac{(d+1)}{2} (2 - A'^2) - \frac{1}{2} A'' \right] \chi_{\tau a} \\ & + \frac{3}{2} A' \partial_a \chi_{\tau\tau} - \partial_a \partial_\tau \chi_{\tau\tau} \\ & + \partial_a \partial_\tau (e^{-A} \text{tr } \chi) - \frac{A'}{2} \partial_a (e^{-A} \text{tr } \chi) \\ & + A' e^{-A} \nabla^b \chi_{ab} - e^{-A} \nabla^c \nabla_c \chi_{\tau a} = 0, \end{aligned} \quad (\text{C.4})$$

$$\begin{aligned} & \partial_\tau^2 \chi_{ab} + \frac{(d-4)}{2} A' \partial_\tau \chi_{ab} + \chi_{ab} \left[ (d+1) - \frac{A'^2}{2} (d-1) - A'' \right] \\ & + \beta_{ab} e^A \left[ -\frac{A'^2}{2} \chi_{\tau\tau} + \frac{A'}{2} \partial_\tau \chi_{\tau\tau} + \chi_{\tau\tau} \right] \\ & + \beta_{ab} \left[ -\frac{A'}{2} e^A \partial_\tau (e^{-A} \text{tr } \chi) - \text{tr } \chi \right] + A' (\nabla_a \chi_{\tau b} + \nabla_b \chi_{\tau a}) \\ & - e^{-A} \nabla^c \nabla_c \chi_{ab} - \nabla_a \nabla_b \chi_{\tau\tau} + e^{-A} \nabla_a \nabla_b \text{tr } \chi = 0. \end{aligned} \quad (\text{C.5})$$

## D $\lambda \neq 0$ , 1 equations on the de Sitter space $dS_{d+1}$

For components  $(\tau\tau)$ ,  $(a\tau)$  and  $(ab)$  of equation (4.17) we find respectively:

$$\begin{aligned} & e^A \partial_\tau^2 (e^{-A} \text{tr } \chi^\lambda) - e^{-A} \nabla^2 \text{tr } \chi^\lambda + 2A' \nabla^a \chi_{\tau a} \\ & + \frac{d}{2} A' \partial_\tau \text{tr } \chi^\lambda + (2 + \lambda(d - \lambda) - \left( d + \frac{1}{2} \right) A'^2) \text{tr } \chi^\lambda = 0, \end{aligned} \quad (\text{D.1})$$

$$\begin{aligned} & \partial_\tau^2 \chi_{\tau a}^\lambda + \left( \frac{d}{2} - 1 \right) A' \partial_\tau \chi_{\tau a}^\lambda + \left( 2 + \lambda(d - \lambda) - \frac{d+1}{2} A'^2 - \frac{1}{2} A'' \right) \chi_{\tau a}^\lambda \\ & - e^{-A} \nabla^2 \chi_{\tau a}^\lambda + A' \partial_a \chi_{\tau\tau}^\lambda + A' e^{-A} \nabla^b \chi_{ba}^\lambda = 0, \end{aligned} \quad (\text{D.2})$$

$$\begin{aligned}
& \partial_\tau^2 \chi_{ab}^\lambda + \left( \frac{d}{2} - 2 \right) A' \partial_\tau \chi_{ab}^\lambda - e^{-A} \nabla^2 \chi_{ab}^\lambda \\
& + \left( 2 + \lambda(d - \lambda) - A'' - \frac{(d-1)}{2} A'^2 \right) \\
& + A' (\nabla_a \chi_{\tau b}^\lambda + \nabla_b \chi_{\tau a}^\lambda) - \frac{1}{2} A'^2 \beta_{ab} \text{tr} \chi^\lambda = 0.
\end{aligned} \tag{D.3}$$

## References

- [1] G. Arcioni and C. Dappiaggi, Exploring the holographic principle in asymptotically flat spacetimes via the BMS group, *Nuclear Phys. B* 674 (2003), 553–592 [arXiv:hep-th/0306142].
- [2] V. Balasubramanian, J. de Boer and D. Minic, Mass, entropy and holography in asymptotically de Sitter spaces, *Phys. Rev. D* 65 (2002), 123508 [arXiv:hep-th/0110108].
- [3] V. Balasubramanian, J. de Boer and D. Minic, Exploring de Sitter space and holography, arXiv:hep-th/0207245.
- [4] R. Beig, Integration of Einstein’s equations near spatial infinity, *Proc. Roy. Soc. London Ser. A* 391 (1984), 295–304.
- [5] R. Beig and B. G. Schmidt, Einstein’s equations near Spatial Infinity, *Comm. Math. Phys.* 87 (1982), 65–80.
- [6] R. Bousso, A Covariant Entropy Conjecture, *J. High Energy Phys.* 07 (1999), 004 [arXiv:hep-th/9905177].
- [7] R. Bousso, A. Maloney and A. Strominger, Conformal vacua and entropy in de Sitter space, *Phys. Rev. D* 65 (2002), 104039 [arXiv:hep-th/0112218].
- [8] J. D. Brown and M. Henneaux, Central charges in the canonical realization of asymptotic symmetries: an example from three dimensional gravity, *Comm. Math. Phys.* 104 (1986), 207–226.
- [9] R. Brustein and A. Yarom, Thermodynamics and area in Minkowski space: Heat capacity of entanglement, *Phys. Rev. D* 69 (2004), 064013 [arXiv:hep-th/0311029].
- [10] J. de Boer and S. N. Solodukhin, A holographic reduction of Minkowski space-time, *Nuclear Phys. B* 665 (2003), 545–593 [arXiv:hep-th/0303006].
- [11] S. de Haro, K. Skenderis and S. N. Solodukhin, Gravity in warped compactifications and the holographic stress tensor, *Classical Quantum Gravity* 18 (2001), 3171–3180 [arXiv:hep-th/0011230].
- [12] S. de Haro, S. N. Solodukhin and K. Skenderis, Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence, *Comm. Math. Phys.* 217 (2001), 595–622 [arXiv:hep-th/0002230].
- [13] S. Deser and A. Waldron, Partial masslessness of higher spins in (A)dS, *Nuclear Phys. B* 607 (2001), 577–604 [arXiv:hep-th/0103198].

- [14] S. Deser and A. Waldron, Null propagation of partially massless higher spins in (A)dS and cosmological constant speculations, *Phys. Lett. B* 513 (2001), 137–141 [arXiv:hep-th/0105181].
- [15] L. Dolan, C. R. Nappi and E. Witten, Conformal operators for partially massless states, *J. High Energy Phys.* 10 (2001), 016 [arXiv:hep-th/0109096].
- [16] A. Erdelyi et al. *Higher transcendental functions*, McGraw-Hill Book Co., New York 1953.
- [17] C. Fefferman and C. R. Graham, Conformal invariants, in: *Élie Cartan et les mathématiques d’aujourd’hui* (Lyon 1984), *Astérisque* (1985), Numéro hors-série, 95–116.
- [18] S. Gubser, I. Klebanov and A. Polyakov, Gauge theory correlators from non-critical string theory, *Phys. Lett. B* 428 (1998), 105–114 [arXiv:hep-th/9802109].
- [19] M. Henningson and K. Skenderis, The holographic Weyl anomaly, *J. High Energy Phys.* 07 (1998), 023 [arXiv:hep-th/9806087]; Holography and the Weyl anomaly, arXiv:hep-th/9812023.
- [20] C. Imbimbo, A. Schwimmer, S. Theisen and S. Yankielowicz, Diffeomorphisms and holographic anomalies, *Classical Quantum Gravity* 17 (1999), 1129–1138 [arXiv:hep-th/9910267].
- [21] K. Krasnov, Twistors, CFT and holography, arXiv:hep-th/0311162.
- [22] F. Loran, Massive spinors and dS/CFT correspondence, arXiv:hep-th/0404135.
- [23] J. Maldacena, The large  $N$  limit of superconformal field theories and supergravity, *Adv. Theor. Math. Phys.* 2 (1998), 231–252 [arXiv:hep-th/9711200].
- [24] R. C. Myers, talk at Black Holes IV, Honey Harbour, May 2003.
- [25] R. C. Myers and M. J. Perry, Black holes in higher dimensional space-times, *Ann. Physics* 172 (1986), 304–347.
- [26] J. Polchinski, S-matrices from AdS spacetime, arXiv:hep-th/9901076.
- [27] I. Sachs and S. N. Solodukhin, Horizon holography, *Phys. Rev. D* 64 (2001), 124023 [arXiv:hep-th/0107173].
- [28] A. Schwimmer and S. Theisen, Diffeomorphisms, anomalies and the Fefferman–Graham ambiguity, *J. High Energy Phys.* 08 (2000), 032 [arXiv:hep-th/0008082].
- [29] A. Schwimmer and S. Theisen, Universal features of holographic anomalies, *J. High Energy Phys.* 10 (2003), 001 [arXiv:hep-th/0309064].
- [30] K. Skenderis and S. N. Solodukhin, Quantum effective action from the AdS/CFT correspondence, *Phys. Lett. B* 432 (2000) 316–322 [arXiv:hep-th/9910023].
- [31] S. N. Solodukhin, Correlation functions of boundary field theory from bulk Green’s functions and phases in the boundary theory, *Nuclear Phys. B* 539 (1999), 403–437 [arXiv:hep-th/9806004].
- [32] M. Spradlin and A. Volovich, Vacuum states and the S-matrix in dS/CFT, *Phys. Rev. D* 65 (2002), 104037 [arXiv:hep-th/0112223].
- [33] A. Strominger, The dS/CFT Correspondence, arXiv:hep-th/0106113.

- [34] L. Susskind, The world as a hologram, *J. Math. Phys.* 36 (1995), 6377–6396 [arXiv:hep-th/9409089].
- [35] L. Susskind, Holography in the flat space limit, arXiv:hep-th/9901079.
- [36] G. 't Hooft, Dimensional Reduction in Quantum Gravity, in: *Salamfestschrift: A Collection of Talks*, World Scientific Series in 20th Century Physics 4, ed. A. Ali, J. Ellis and S. Randjbar-Daemi, World Scientific, Singapore 1993, gr-qc/9310026.
- [37] E. Witten, Anti de Sitter space and holography, *Adv. Theor. Math. Phys.* 2 (1998), 253–291 [arXiv:hep-th/9802150].
- [38] E. Witten, Quantum gravity in de Sitter space, arXiv:hep-th/0106109.



# Non-trivial, static, geodesically complete space-times with a negative cosmological constant II. $n \geq 5$

*Michael T. Anderson\*, Piotr T. Chruściel\*\* and Erwann Delay‡*

*Department of Mathematics, S.U.N.Y. at Stony Brook  
Stony Brook, N.Y. 11794-3651, U.S.A.  
email: anderson@math.sunysb.edu*

*Département de Mathématiques, Faculté des Sciences  
Parc de Grandmont, 37200 Tours, France  
email: piotr.chrusciel@lmtp.univ-tours.fr*

*Département de Mathématiques, Faculté des Sciences  
rue Louis Pasteur, 84000 Avignon, France  
email: erwann.delay@univ-avignon.fr*

**Abstract.** We show that the recent work of Lee [24] implies existence of a large class of new singularity-free strictly static Lorentzian vacuum solutions of the Einstein equations with a negative cosmological constant. This holds in all space-time dimensions greater than or equal to four, and leads both to strictly static solutions and to black hole solutions. The construction allows in principle for metrics (whether black hole or not) with Yang–Mills-dilaton fields interacting with gravity through a Kaluza–Klein coupling.

## 1 Introduction

In recent work [3] we have constructed a large class of non-trivial static, geodesically complete, four-dimensional vacuum space-times with a negative cosmological constant. The object of this paper is to establish existence of higher dimensional analogues of the above.

---

\*Partially supported by an NSF Grant DMS 0305865.

\*\*Partially supported by a Polish Research Committee grant 2 P03B 073 24.

‡Partially supported by the ACI program of the French Ministry of Research.

More precisely, we wish to show that for  $\Lambda < 0$  and  $n \geq 4$  there exist  $n$ -dimensional *strictly static*<sup>1</sup> solutions  $(\mathcal{M}, g)$  of the vacuum Einstein equations with the following properties:

- (1)  $(\mathcal{M}, g)$  is diffeomorphic to  $\mathbb{R} \times \Sigma$ , for some  $(n - 1)$ -dimensional spacelike Cauchy surface  $\Sigma$ , with the  $\mathbb{R}$  factor corresponding to the action of the isometry group.
- (2)  $(\Sigma, g_\Sigma)$ , where  $g_\Sigma$  is the metric induced by  $g$  on  $\Sigma$ , is a complete Riemannian manifold.
- (3)  $(\mathcal{M}, g)$  is geodesically complete.
- (4) All polynomial invariants of  $g$  constructed using the curvature tensor and its derivatives up to any finite order are bounded on  $\mathcal{M}$ .
- (5)  $(\mathcal{M}, g)$  admits a globally hyperbolic (in the sense of manifolds with boundary) conformal completion with a timelike  $\mathcal{I}$ . The completion is smooth if  $n$  is even, and is of differentiability class at least  $C^{n-2}$  if  $n$  is odd.
- (6)  $(\Sigma, g_\Sigma)$  is a conformally compactifiable manifold, with the same differentiability as in point 5.
- (7) The connected component of the group of isometries of  $(\mathcal{M}, g)$  is exactly  $\mathbb{R}$ , with an associated Killing vector  $X$  being timelike throughout  $\mathcal{M}$ .
- (8) There exist no local solutions of the Killing equation other than the (globally defined) timelike Killing vector field  $X$ .

An example of a manifold satisfying points 1–6 above is of course  $n$ -dimensional anti-de Sitter solution. Clearly it *does not* satisfy points 7 and 8.

We expect that there exist stationary *and not static* solutions as above, which can be constructed by solving an asymptotic Dirichlet problem for the Einstein equations in a conformally compactifiable setting. We are planning to study this question in the future.

In a black hole context we have an obvious variation of the above; we discuss this in more detail in Section 2.3.

Throughout this work we restrict attention to dimension  $n \geq 4$ .

Our approach is, in some sense, opposite to that in [25], [28], where techniques previously used in general relativity have been employed to obtain uniqueness results in a Riemannian setting. Here we start with Riemannian Einstein metrics and obtain Lorentzian ones by “Wick rotation”, as follows: Suppose that  $(M, g)$  is an  $n$ -dimensional conformally compactifiable Einstein manifold of the form  $M = \Sigma \times S^1$ , and that  $S^1$  acts on  $M$  by rotations of the  $S^1$  factor while preserving the metric. Denote by  $X = \partial_\tau$  the associated Killing vector field, and assume that  $X$  is orthogonal to the

---

<sup>1</sup>We shall say that a space-time is *strictly static* if it contains a globally timelike hypersurface orthogonal Killing vector field.

sets  $\Sigma \times \{\exp(i\tau)\}$ , where  $\exp(i\tau) \in S^1 \subset \mathbb{C}$ . Then the metric  $g$  can be (globally) written in the form

$$g = u^2 d\tau^2 + g_\Sigma, \quad \mathcal{L}_X u = \mathcal{L}_X g_\Sigma = g_\Sigma(X, \cdot) = 0. \quad (1.1)$$

It is straightforward to check that the space-time  $(\mathcal{M} := \mathbb{R} \times \Sigma, \mathfrak{g})$ , with

$$\mathfrak{g} = -u^2 dt^2 + g_\Sigma \quad (1.2)$$

is a static solution of the vacuum Einstein equations with negative cosmological constant, with Killing vector field  $\partial_t$ .

In order to continue, some definitions are in order: Let  $M$  be the interior of a smooth, compact,  $n$ -dimensional manifold-with-boundary  $\bar{M}$ . A Riemannian manifold  $(M, g)$  will be said to be *conformally compact at infinity*, or *conformally compact*, if

$$g = x^{-2} \bar{g},$$

for a smooth function  $x$  on  $\bar{M}$  such that  $x$  vanishes precisely on the boundary  $\partial \bar{M}$  of  $\bar{M}$ , with non-vanishing gradient there. Further  $\bar{g}$  is a Riemannian metric which is regular up-to-boundary on  $\bar{M}$ ; the differentiability properties near  $\partial M$  of a conformally compact metric  $g$  will always refer to those of  $\bar{g}$ . The operator

$$P := \Delta_L + 2(n-1),$$

where  $\Delta_L$  is the Lichnerowicz Laplacian (cf., e.g., [24]) associated with  $g$ , plays an important role in the study of such metrics. An Einstein metric  $g$  with scalar curvature  $-n(n-1)$  will be said *non-degenerate* if  $P$  has trivial  $L^2$  kernel on the space of trace-free symmetric 2-tensors. We prove the following openness theorem around static metrics for which the Killing vector field has no zeros (see Section 3.1 for terminology):

**Theorem 1.1.** *Let  $(M, g)$  be a non-degenerate, strictly globally static conformally compact Riemannian Einstein metric, with conformal infinity  $\gamma := [\bar{g}|_{\partial M}]$  (conformal equivalence class). Then any small static perturbation of  $\gamma$  is the conformal infinity of a strictly globally static Riemannian Einstein metric on  $M$ .*

Theorem 1.1 is established by the arguments presented at the beginning of Section 2.2, compare [3] for a more detailed treatment. In Section 2.2 we also describe a subclass of Riemannian metrics given by Theorem 1.1 that leads to Lorentzian Einstein metrics with the properties 1–8 listed above. In particular we show that our construction provides non-trivial solutions near the AdS solution in all dimensions.

By an abuse of terminology, Riemannian solutions for which the set of zeros of the Killing vector field  $X$  contains an  $n-2$  dimensional surface  $N \equiv N^{n-2}$  will be referred to as *black hole solutions*;  $N$  will be called<sup>2</sup> *the horizon*. The corresponding openness result here reads:

---

<sup>2</sup>In the corresponding Lorentzian solution the set  $N$  will be the pointwise equivalent of the *black hole bifurcation surface*, i.e., the intersection of the past and future event horizons.



**Theorem 1.2.** *Let  $(M, g)$  be a non-degenerate, globally static conformally compact Riemannian Einstein metric, with strictly globally static conformal infinity  $\gamma$ . Suppose that*

- (1) *either  $M = N \times \mathbb{R}^2$ , with the action of  $S^1$  being by rotations of  $\mathbb{R}^2$ , or*
- (2)  *$H_{n-3}(M) = \{0\}$ , and the zero set of  $X$  is a smooth  $(n-2)$ -dimensional submanifold with trivial normal bundle.*

*Then any small globally static perturbation of  $\gamma$  is the conformal infinity of a static black hole solution with horizon  $N$ .*

The proof of Theorem 1.2 is given at the end of Section 2.3.

This paper is organised as follows: In Section 2.1 we review some results concerning the Riemannian equivalent of the problem at hand. In Section 2.2 we sketch the construction of the new solutions, and we show that our results in the remaining sections prove existence of non-trivial solutions which are near the  $n$ -dimensional anti-de Sitter one. In Sections 2.3 and 2.4 we discuss how the method here can be used to produce solutions with black holes, and with Kaluza–Klein type coupling to matter. In Section 3 we give conditions under which hypersurface-orthogonality descends from the boundary to the interior: this is done under the hypothesis of *topological staticity* in Section 3.1, and under the hypothesis of *existence of a twist potential* in Section 3.2. Appendix A studies the action of Killing vector fields near the conformal boundary, and contains results about extendibility of conformal isometries of  $\partial M$  to isometries of  $M$ . In Appendix B we derive the norm and twist equations for Einstein metrics in all dimensions; those equations are of course well known in dimension four. In Appendix C we study the structure of the metric near fixed points of the action of the isometry group, as needed for the analysis of Section 3. In Appendix D we calculate the sectional curvatures of  $n$ -dimensional Kottler-type solutions, proving in particular the existence of a large family of non-degenerate  $n$ -dimensional black hole solutions, for any  $n$ .

## 2 The solutions

We start by a review of the associated Riemannian problem.

### 2.1 The Riemannian solutions

An important result on the structure of conformally compact Einstein manifolds is the following, which is an improvement of earlier results in [8], [21], (cf. also [1], [2], [15] for related or similar results):

**Theorem 2.1** (Lee [24]). *Let  $M$  be the interior of a smooth, compact,  $n$ -dimensional manifold-with-boundary  $\bar{M}$ ,  $n \geq 4$ , and let  $g_0$  be a non-degenerate Einstein metric on*

$M$  that is conformally compact of class  $C^{l,\beta}$  with  $2 \leq l \leq n-2$  and  $0 < \beta < 1$ . Let  $\rho$  be a smooth defining function for  $\partial M$ , and let  $\gamma_0 = \rho^2 g_0|_{\partial M}$ . Then there is a constant  $\epsilon > 0$  such that for any  $C^{l,\beta}$  Riemannian metric  $\gamma$  on  $\partial M$  with  $\|\gamma - \gamma_0\|_{C^{l,\beta}} < \epsilon$ , there exists an Einstein metric  $g$  on  $M$  that has  $[\gamma]$  as conformal infinity and is conformally compact of class  $C^{l,\beta}$ .

The non-degeneracy condition above will hold e.g. in the following circumstances:

**Theorem 2.2.** *Under the remaining hypotheses of Theorem 2.1,  $\Delta_L + 2(n-1)$  has trivial  $L^2$  kernel on the space of trace-free symmetric 2-tensors if either of the following hypotheses is satisfied:*

- (a) *At each point, either all the sectional curvatures of  $g_0$  are nonpositive, or all are bounded below by  $-2(n-1)/n$ .*
- (b) *The Yamabe invariant of  $[\gamma]$  is nonnegative and  $g_0$  has sectional curvatures bounded above by  $(n-1)(n-9)/8(n-2)$ .*

Here  $g_0$  has been normalised so that its scalar curvature equals  $-n(n-1)$ .

Point (b) of Theorem 2.2 is due to Lee [24]. Point (a) is easily inferred from Lee's arguments, for completeness we give the proof in Appendix D.

The regularity of the solutions above can be improved as follows [12]; the following result is the only exception to the rule that  $n \geq 4$  in this paper:

**Theorem 2.3.** *Let  $g$  be a  $C^2$  conformally compactifiable Einstein metric on an  $n$ -dimensional manifold  $M$  with  $C^\infty$  smooth boundary metric  $[\gamma]$ ,  $n \geq 3$ .*

- (1) *If  $n$  is even or equal to three, then there exists a differentiable structure on  $\bar{M}$  such that  $g$  is smoothly compactifiable.*
- (2) *If  $n$  is odd, then there exist local coordinates  $(x, v^C)$  near the boundary so that*

$$g = x^{-2}(dx^2 + \gamma_{AB}dv^A dv^B), \quad (2.1)$$

with the functions  $\gamma_{AB}$  of the form

$$\gamma_{AB}(x, v^C) = \phi_{AB}(x, v^C, x^{n-1} \ln x), \quad (2.2)$$

with  $\phi_{AB}(x, v^C, z)$  smooth functions of all their arguments. Further, there exists a differentiable structure on  $\bar{M}$  so that  $g$  is smoothly compactifiable if and only if  $(\partial_z \phi_{AB})(0, v^C, 0)$  vanishes.

Explicit formulae for  $(\partial_z \phi_{AB})(0, v^C, 0)$  in low dimensions can be found in [22]. It follows from the results in [16] that  $(\partial_z \phi_{AB})(0, v^C, 0) = 0$  when  $\gamma(0)$  has a representative which is Einstein, so that the filling metric  $g$  is smoothly compactifiable in this case, independently of the dimension.

## 2.2 From Riemannian to Lorentzian solutions

In order to implement the procedure leading from (1.1) to (1.2), we start by solving the Einstein equation for  $g$  with a prescribed conformal infinity  $[\gamma]$  on  $\partial M = \partial \Sigma \times S^1$ , using e.g. Theorem 2.1. One further assumes that rotations of the  $S^1$  factor are conformal isometries of  $[\gamma]$ . In order to carry through the construction one needs to know that conformal isometries of  $[\gamma]$  extend to isometries of  $g$ . This is proved in [1] in dimension four without further restrictions, and under a non-degeneracy condition in higher dimensions; we give an alternative proof of this fact in Appendix A. A somewhat weaker version of the extension result in Appendix A has been independently proved, using essentially the same argument, by Wang [29]; compare [27] for yet another independent similar result. We include the details of our proof, because in the course thereof we derive some properties of Killing vector fields which are used elsewhere in the paper. The final step is to ensure hypersurface-orthogonality of the  $U(1)$  action. This is done in the next section.

We refer the reader to [3] for a detailed analysis of the four-dimensional case, where considerably stronger results are available. However, even in dimension four the results on hypersurface-orthogonality in Section 3.2 do not follow from those in [3].

In order to show that the intersection of the set of hypotheses of our results below is not empty, let  $(M, g_0)$  be the Riemannian equivalent of the  $n$ -dimensional anti-de Sitter space-time, with  $g_0$  obtained by reversing the procedure described above. Thus,  $M$  is diffeomorphic to  $B^{n-1} \times S^1$ , where  $B^{n-1}$  is the  $(n-1)$ -dimensional open unit ball. Let  $\alpha$  be any strictly positive function on the unit  $(n-2)$ -dimensional sphere  $S^{n-2}$ , and consider the following metric  $\gamma$  on  $S^{n-2} \times S^1$ :

$$\gamma = \alpha^2 d\varphi^2 + h, \quad \mathcal{L}_{\partial_\varphi} h = h(\partial_\varphi, \cdot) = 0. \quad (2.3)$$

Here  $\varphi$  is the coordinate along the  $S^1$  factor of  $\partial M$ . Since  $g_0$  has negative sectional curvatures, by Lee's Theorem 2.2 there exists an Einstein metric on  $M$  with conformal infinity  $[\gamma]$  provided that  $\alpha$  is sufficiently close to one and  $h$  is sufficiently close to the unit round metric.<sup>3</sup> By Proposition A.1  $\partial_\varphi$  extends to a Killing vector field  $X$  on  $M$ . Since the corresponding Killing vector field in  $(M, g_0)$  did not have any zeros, continuous dependence of solutions of (A.14) upon the metric implies that the same will hold for  $X$  (making  $\alpha$  closer to 1 and  $h$  closer to the unit round metric if necessary). In fact, this also shows that the orbit space  $M/S^1$  will be a smooth manifold, diffeomorphic to  $B^{n-1}$ . The fact that  $H_{n-3}(B^{n-1}) = \{0\}$  implies existence of the twist potential  $\tau$ , and since the boundary action is hypersurface orthogonal we can use Theorem 3.3 to obtain hypersurface-orthogonality throughout  $M$ . Therefore the Riemannian solutions so obtained lead to Lorentzian equivalents, as described above.

---

<sup>3</sup>For definiteness we only consider metrics close to anti-de Sitter, though an identical argument can be used whenever the results of [1], or those of [24], apply. In particular, in view of the results of [1], in dimension four the construction described here applies in much larger generality.

Let us justify the claims made in the Introduction. Point 1 follows immediately from the discussion around Equation (1.1). Point 6 follows from Theorem 2.3 and from what is said in the proof of Theorem 3.3. Point 2 is a straightforward corollary of point 6. Point 4 follows from well known properties of conformally compactifiable metrics. The geodesic completeness of the static metrics so obtained has been proved in [3, Section 4]. Global hyperbolicity in point 5 is established in the course of the proof of Theorem 4.1 of [13], while the differentiability properties claimed in point 5 follow from point 6. The argument given at the end of [3, Section 4] gives non-existence of other global or local Killing vector fields when the boundary metric has no other conformal isometries.

### 2.3 Black hole solutions

Let us pass to a discussion of static black hole solutions in higher dimensions. The standard examples of static Riemannian AdS-type black hole solutions are on the manifold  $M = N^{n-2} \times \mathbb{R}^2$ , with  $N := N^{n-2}$  compact, and with metric of form:

$$g_m = V^{-1}dr^2 + Vd\theta^2 + r^2g_N, \quad (2.4)$$

where  $g_N$  is any Einstein metric,  $\text{Ric}_{g_N} = \lambda g_N$ , with  $g_N$  scaled so that  $\lambda = \pm(n-3)$  or 0. Then for  $V = V(r)$  given by

$$V = c + r^2 - (2m)/r^{n-3}, \quad (2.5)$$

with  $c = \pm 1$  or 0 respectively,  $g_m$  is static Einstein, with  $\text{Ric}_{g_m} = -(n-1)g_m$ . These are just the analogues of Kottler metrics in higher dimensions. The length of  $S^1 \ni \theta$  is determined by  $m$  together with the requirement that  $g_m$  be a smooth metric at the “horizon”  $r = r_0$ , the largest root of  $V(r)$ . (This restriction of course disappears in the Lorentzian setting).

Now each such  $g_m$  belongs to a 1-parameter family of metrics, parameterised by the mass  $m$ . One expects that for generic values of  $m$ , the metric  $g_m$  is non-degenerate, as defined in the introduction. In fact, when  $n = 4$ , the AdS–Schwarzschild metric has non-trivial kernel exactly for one specific value of  $m$ ,<sup>4</sup> while the toroidal black holes, as well as the higher genus Kottler black holes, are always non-degenerate. Those last two results are well known, and in any case are proved in Appendix D, where we also show existence of a large class of non-degenerate black hole solutions for all  $n \geq 5$ .

Assuming non-degeneracy, suppose we then consider local perturbations of the conformal infinity, preserving the static structure at infinity – so, e.g., just vary the function, say  $\alpha$ , which describes the length of the  $S^1$ ’s at infinity, keeping the remainder of the conformal boundary metric fixed. Then by the results in Appendix A, we get extension of the isometric  $S^1$  action on the (locally unique) Einstein filling metric of

---

<sup>4</sup>In dimension four it is known that all continuous isometries descend from the boundary to the interior [1]. When infinity is spherical, the elements of the kernel have to be spherically symmetric, and one can conclude using the (generalised) Birkhoff theorem.

Theorem 2.1. We need to prove the extension is static also, this will follow if we can use Theorems 3.1 or 3.4. For the former, we can use the fact that *topological staticity*, as defined at the beginning of Section 3.1, is stable under continuous deformations of the metric (compare [3, Lemma 2.6]; the restriction  $\dim M = 4$  there is not needed). This proves Theorem 1.2 under the hypothesis that the action of  $S^1$  is by rotations of  $\mathbb{R}^2$ , as that action is topologically static.

Another condition which can lead to staticity is  $H_{n-3}(M) = 0$ , for then we have existence of the twist potential. After a small change of  $\gamma$ , the  $S^1$  action is again a small perturbation of the original static  $S^1$  action. This means that the only fixed point set of the  $S^1$  action (the zero set of  $X$ ), is again a smooth  $(n - 2)$  manifold, say  $N$ , with the normal  $S^1$  bundle remaining trivial. Hence, condition (3.20) of Theorem 3.4 is satisfied, and staticity follows.

## 2.4 Kaluza–Klein solutions

Similarly one should be able to construct space-times as described in the introduction, with or without black hole regions, which satisfy the Einstein–Yang–Mills–dilaton field equations with a Kaluza–Klein coupling [14]; solutions belonging to this family have been numerically constructed in [9], [30]. More precisely, suppose that one has a conformally compactifiable Einstein manifold  $(M^n, g)$  satisfying the following: a)  $\partial M^n = S^1 \times M^{n-2}$ ; with a product boundary conformal class  $[\tilde{g}|_{\partial M^n}]$  such that b) rotations along  $S^1$  are conformal isometries; c)  $(M^n, g)$  satisfies the hypotheses of Theorem 2.1 (compare Theorem 2.2); d) the  $S^1$  action on  $M$  associated with the rotations of  $S^1$  on  $\partial M$  satisfies the hypotheses of one of the Theorems 3.1, 3.3, 3.4 or 3.6. Then any connected Lie group  $G$  of conformal isometries of  $[\tilde{g}|_{M^{n-2}}]$  with a free action will then lead to a Kaluza–Klein type Yang–Mills gauge group for the associated Lorentzian solutions.

## 3 Hypersurface-orthogonality

We use the notations of Appendix A. We wish to show that  $X$  is hypersurface-orthogonal; clearly a necessary condition for that is that  $\hat{X}(0)$  be hypersurface orthogonal. We suspect that this condition is sufficient, but we have not been able to prove that. In dimension four hypersurface-orthogonality has been proved in [3] under the hypothesis of *topological staticity* of  $X$ , as defined there, cf. below. We shall show in Section 3.1 that the result generalises to higher dimensions. We also give an alternative approach, assuming that we have the *twist potential* at our disposal.

In what follows we will assume that  $\hat{X}(0)$  arises from an  $S^1$  action on  $\partial M$ ; this hypothesis can be replaced by the existence of a hypersurface  $\mathcal{S}$  in  $M \setminus \{g(X, X) = 0\}$  which is transversal to  $X$  – identical proofs apply, with  $\Sigma$  there replaced by  $\mathcal{S}$ .

Before proceeding further we have to introduce some notation. Let  $\Sigma$  be the orbit space of the  $S^1$  action, and let  $\mathring{\Sigma}$  denote the set of orbits of principal type, then  $\mathring{\Sigma}$  is a smooth manifold forming an open dense subset of  $\Sigma$ . We set

$$\mathring{M} := (\pi_\Sigma^*)^{-1} \mathring{\Sigma}.$$

Let  $g_\Sigma$  be the induced metric on  $\mathring{\Sigma}$ ; thus the metric  $g$  has the form

$$g = u^2(d\phi + \theta)^2 + \pi_\Sigma^* g_\Sigma, \quad (3.1)$$

where  $\theta$  is a connection 1-form,  $u$  is the length of the Killing field  $X = \partial/\partial\phi$ , and  $\pi_\Sigma^*: M \rightarrow \Sigma$  is the canonical projection. The parameter  $\phi$  parameterises a circle  $S^1$ . The space  $\Sigma$  is in general a  $(n-1)$ -orbifold; there may be stratified submanifolds in  $\Sigma$  along which the metric has cone singularities. Of course  $\Sigma$  is non-compact – it has a boundary at infinity  $\partial_\infty \Sigma$  corresponding to the orbit space of the  $S^1$  action on  $\partial M$ . Redefining the  $S^1$  action if necessary, one can without loss of generality assume that the action is free on  $(\pi_\Sigma^*)^{-1} \mathring{\Sigma}$ . The set of non-principal orbits is the union of trivial orbits  $\Sigma_{\text{sing},0}$  which correspond to fixed points of the action, and of special orbits  $\Sigma_{\text{sing},\text{iso}}$  which are circles with non-trivial isotropy group:

$$\Sigma_{\text{sing}} := \Sigma_{\text{sing},0} \cup \Sigma_{\text{sing},\text{iso}}, \quad M_0 := (\pi_\Sigma^*)^{-1} \Sigma_{\text{sing},0} = \{u = 0\} \subset M.$$

In dimension four the fixed point set consists of isolated points and smooth, totally geodesic submanifolds, those have been called “nuts” and “bolts”; nut fixed points are isolated points in  $M$ , while the bolts correspond to totally geodesic surfaces in  $\partial \mathring{\Sigma}$ , cf. [20]. The structure of the orbits near fixed points is discussed in all dimensions in Appendix C, see also [17], [18]. Off the fixed point set the isotropy group is finite and so the orbits are circles. The isotropy group may change. For example, in dimension three one can have  $S^1$  actions on a solid torus  $D^2 \times S^1$  which are free on  $(D^2 \setminus \{0\}) \times S^1$ , with non-trivial isotropy on the core curve  $\{0\} \times S^1$ . One can take such and product with  $S^1$  again to obtain higher dimensional manifolds which have (isolated) curves where the isotropy jumps up [10], [17], [18].

Let the *twist*  $(n-3)$ -form  $\hat{\omega}$  be defined on  $M$  by the equation (the definition here differs by a factor of 2 from the definition in [3])

$$\hat{\omega} = *_g(\xi \wedge d\xi), \quad \xi := g(X, \cdot),$$

where  $*_g$  is the Hodge duality operator with respect to the metric  $g$ . On  $\mathring{M}$  the form  $\hat{\omega}$  is the pull-back by  $\pi_\Sigma$  of a form  $\omega$  defined on  $\mathring{\Sigma}$ . It is well known in dimension four, and it is shown in general in Appendix B, that  $\omega$  (and hence  $\hat{\omega}$ ) is closed.

### 3.1 Topologically static actions

We will use the following terminology, as in [3]. The  $S^1$  action on  $(M, g)$  is *strictly globally static* if  $(M, g)$  is globally a warped product of the form

$$M = S^1 \times \Sigma, \quad g = u^2 d\phi^2 + \pi_\Sigma^* g_\Sigma, \quad (3.2)$$

where  $u: \Sigma \rightarrow \mathbb{R}$  is strictly positive and  $g_\Sigma$  is a complete metric on  $\Sigma$ ,  $\partial\Sigma = \emptyset$ . In this case, the  $S^1$  action is just given by rotations in the  $S^1$  factor. The  $S^1$  action is *globally static* if (3.2) holds with  $u = 0$  somewhere. In this case, the locus  $\{u = 0\}$  is not empty, but there are no exceptional orbits. Next, the  $S^1$  action is *topologically static* if the  $S^1$  bundle  $S^1 \rightarrow P \rightarrow \Sigma_P$  is a trivial bundle, i.e. it admits a section. (Here  $P$  is the union of principal orbits, while we use the symbol  $E$  for the union of exceptional ones.) This is equivalent to the existence of a cross-section of the  $S^1$  fibration  $P \cup E \rightarrow \Sigma_{P \cup E}$ .<sup>5</sup> Finally, we define the  $S^1$  action to be *locally static* if every point of  $(M, g)$  has a neighborhood isometric to a neighborhood of a point with metric of the form (3.2); this is equivalent to the usual notion of static in the sense of the existence of a hypersurface orthogonal Killing field.

We shall use an obvious equivalent of the above for an  $\mathbb{R}$  action by isometries on a Lorentzian manifold  $(\mathcal{M}, g)$ , with the further restriction that the associated Killing vector field be timelike almost everywhere.

The main result of this section is the following:

**Theorem 3.1.** *Let  $(M, g)$  be a smoothly compactifiable Einstein metric on  $M$  with  $\dim M \geq 4$ . Suppose the free  $S^1$  action at conformal infinity  $(\partial M, \gamma)$  is strictly globally static, i.e.*

$$\partial M = S^1 \times V, \quad (3.3)$$

*and the  $S^1$  action on  $(M, g)$  is topologically static.*

*Then the  $S^1$  action on  $(M, g)$  is locally static, i.e.  $(M, g)$  is locally of the form (3.2) (with  $\{u = 0\} \neq \emptyset$  possibly).*

*Proof.* The method of proof is identical to that in [3]. As pointed out in [27], regardless of dimension and signature one has the identity<sup>6</sup>

$$d\left(\frac{\xi \wedge \hat{\omega}}{u^2}\right) = \pm \frac{|\hat{\omega}|^2 i_X(\text{Vol})}{u^4}, \quad (3.4)$$

with the sign  $\pm$  being determined by the signature of the metric. Here  $\text{Vol}$  is the volume form. Integrating (3.4) over any cross-section  $\Sigma$  one will obtain  $\hat{\omega} = 0$  provided that the boundary term arising from the left-hand side of (3.4) vanishes. This requires sufficiently fast fall-off of  $\hat{\omega}$  near the conformal infinity  $\partial M$ , which is provided by the following Lemma. The hypotheses of Theorem 3.1 imply that the Killing vector has no zeros on  $\partial M$ , but we do not need this assumption for the proof that follows:

**Lemma 3.2.** *In the coordinate system of (A.1) we have*

$$\hat{\omega}_{rA_1 \dots A_{n-4}} = O(r), \quad \hat{\omega}_{AA_1 \dots A_{n-4}} = O(1),$$

*with  $\hat{\omega}_{rA_1 \dots A_{n-4}} = \omega_{rA_1 \dots A_{n-4}}$  and  $\hat{\omega}_{AA_1 \dots A_{n-4}} = \omega_{AA_1 \dots A_{n-4}}$ .*

<sup>5</sup> $S$  will be called a cross-section of a fibration if  $S$  meets every fiber at least once, with the intersection being transverse.

<sup>6</sup>The identity here differs by a factor of 2 from the one in [3] because the form  $\omega$  here is twice that in [3].

*Proof.* The idea of the proof is to use the equation

$$\nabla^k \nabla_k X_i = -\text{Ric}(g)_i^j X_j,$$

together with the fact that  $g$  is Einstein, to obtain more information about the decay of the relevant components of the metric. We work in the coordinate system of Appendix A, and use the conventions there. We have (recall that  $\hat{X}_r \equiv 0$  by Remark A.3)

$$\begin{aligned} 2\nabla^k \nabla_k X_A &= \nabla^k (\nabla_k X_A - \nabla_A X_k) \\ &= \nabla^k (\partial_k X_A - \partial_A X_k) \\ &= r^2 \{ \partial_r^2 X_A - \Gamma_{rr}^r \partial_r X_A - \Gamma_{rA}^E \partial_r X_E \\ &\quad + \hat{g}^{EF} [\partial_E \partial_F X_A - \partial_E \partial_A X_F - \Gamma_{EF}^r \partial_r X_A - \Gamma_{EF}^C (\partial_C X_A - \partial_A X_C) \\ &\quad + \Gamma_{EA}^r \partial_r X_F - \Gamma_{EA}^C (\partial_F X_C - \partial_C X_F)] \}. \end{aligned} \quad (3.5)$$

Consider any point  $p$  on the conformal boundary at which  $\hat{X}(0)$  does not vanish. As  $\hat{X}(0)$  is hypersurface orthogonal, we can choose a local coordinate system on the boundary at infinity, defined on a neighborhood of  $p$ , such that  $x^A = (\varphi, x^a)$  ( $a = 3, \dots, n$ ), with  $\hat{X}(0) = \partial_\varphi$  and

$$\hat{g}(0)_{AB} dx^A dx^B = \hat{g}(0)_{\varphi\varphi} d\varphi^2 + \hat{g}(0)_{ab} dx^a dx^b.$$

Recall that  $X_A = r^{-2} \hat{g}_{AB} \hat{X}^B = r^{-2} \hat{g}_{A\varphi}$ . From (A.2)–(A.3) and (3.5) we then have

$$\begin{aligned} 2\nabla^k \nabla_k X_A &= 2(n-1)r^{-2} \hat{g}_{A\varphi} - nr^{-1} \hat{g}'_{A\varphi} + r^{-1} \hat{g}^{CD} (2\hat{g}'_{DA} \hat{g}_{C\varphi} - \hat{g}'_{CD} \hat{g}_{A\varphi}) \\ &\quad + \hat{g}''_{A\varphi} + \frac{1}{2} \hat{g}^{CD} (-2\hat{g}'_{DA} \hat{g}'_{C\varphi} + \hat{g}'_{CD} \hat{g}'_{A\varphi}) + 2\hat{\nabla}^E \hat{\nabla}_E \hat{X}_A, \end{aligned}$$

where a prime denotes an  $r$ -derivative. From equation (A.10) we have

$$\hat{g}'_{A\varphi} = \hat{g}^{CD} \hat{g}'_{DA} \hat{g}_{C\varphi}$$

and then as  $\text{Ric}(g) = -(n-1)g$ ,

$$\begin{aligned} (2-n)r^{-1} \hat{g}'_{A\varphi} + \hat{g}''_{A\varphi} \\ = r^{-1} \hat{g}^{CD} \hat{g}'_{CD} \hat{g}_{A\varphi} - \frac{1}{2} \hat{g}^{CD} (-2\hat{g}'_{DA} \hat{g}'_{C\varphi} + \hat{g}'_{CD} \hat{g}'_{A\varphi}) - 2\hat{\nabla}^E \hat{\nabla}_E \hat{X}_A. \end{aligned}$$

Consider that equation when  $A = a$ ; straightforward but tedious algebra shows that its right-hand side can be written as a linear combination of the  $\hat{g}_{b\varphi}$ 's together with their first derivatives and their second  $\partial_c$ -derivatives, with bounded, differentiable up-to-boundary, coefficients built out of the  $g_{AB}$ 's and their derivatives. (For example,

$$\hat{g}^{CD} \hat{g}'_{Da} \hat{g}'_{C\varphi} = \hat{g}^{\varphi\varphi} \hat{g}'_{\varphi a} \hat{g}'_{\varphi\varphi} + \hat{g}^{c\varphi} \hat{g}'_{\varphi a} \hat{g}'_{c\varphi} + \hat{g}^{c\varphi} \hat{g}'_{ca} \hat{g}'_{\varphi\varphi} + \hat{g}^{cd} \hat{g}'_{da} \hat{g}'_{\varphi\varphi},$$

and note that  $\hat{g}^{c\varphi}$  is a rational function involving the  $g_{b\varphi}$ 's which vanishes when the latter do, hence can be written as an expression linear in the  $g_{b\varphi}$ 's with coefficients which depend upon the  $g_{AB}$ 's.) Then as  $\hat{g}_{a\varphi}(0) = 0$  we have that  $\hat{g}_{a\varphi} = O(r^2)$ .



Taylor expanding and matching coefficients in front of powers of  $r$  one is led to

$$\hat{g}_{a\varphi} = O(r^{n-1}). \quad (3.6)$$

Set

$$\mu^2 := \hat{g}(0)(\hat{X}(0), \hat{X}(0)).$$

Since  $g$  is Einstein we have [16]  $\hat{g}(r) = \hat{g}(0) + O(r^2)$ , so that

$$u^2 := g(X, X) = r^{-2}\hat{g}(\hat{X}, \hat{X}) = r^{-2}(\mu^2 + O(r^2)). \quad (3.7)$$

In particular  $u$  behaves as  $1/r$  (recall that we are so far working away from the zero set of  $\hat{X}(0)$ ). Consider the two-form  $\hat{\lambda}$  defined in (B.9); (3.6) gives

$$\hat{\lambda} = \sum_a O(r^{n-5})dr \wedge dx^a + \sum_{a,b} O(r^{n-4})dx^a \wedge dx^b. \quad (3.8)$$

The coordinates  $(r, x^a)$  can be used as local coordinates on the quotient manifold  $\Sigma$ , in those coordinates  $\hat{\lambda}_{ab} = \lambda_{ab}$ ,  $\hat{\lambda}_{ar} = \lambda_{ar}$ ,  $\hat{\lambda}_{i\varphi} = 0$ . It follows now that in any coordinate system  $\{x^A\}$  on the conformal boundary as in (A.1), not necessarily adapted to the hypersurface-orthogonal character of  $X$ , we will have

$$\lambda_{AB} = O(r^{n-4}), \quad \lambda_{Ar} = O(r^{n-5}), \quad (3.9)$$

as long as we are not at a point at which  $\hat{X}(0)$  vanishes. Now,  $u\hat{\lambda}$  is defined and smooth regardless of zeros of  $X$ , which implies that (3.9) holds globally on each domain of definition of the coordinates  $x^A$  independently of the existence of zeros of  $\hat{X}(0)$  there. We finally obtain

$$\lambda_{AB} = O(r^{n-4}) \iff \lambda^{AB} = g^{AC}g^{BD}\lambda_{CD} = O(r^n) \quad (3.10)$$

with a similar equivalence for  $\lambda_{Ar}$ . Let  $\eta_{A_1 \dots A_{n-2}}$  be totally anti-symmetric, equal to 1 if  $A_1 \dots A_{n-2}$  is an even permutation of  $1, 2, \dots, n-2$ . It is clear from (A.12)–(A.13) that

$$\sqrt{\det g_\Sigma} = r^{-(n-1)}\sqrt{\det \hat{g}_\Sigma(r)},$$

where  $\hat{g}_\Sigma(r)$  is the metric on the quotient  $(\{r = \text{const}\}, \hat{g}(r))/S^1$ . Choosing a convenient orientation, from the definition (B.16) of  $\omega$  we have

$$\begin{aligned} \omega_{rA_1 \dots A_{n-4}} &= \sqrt{\det g_\Sigma} \eta_{A_1 \dots A_{n-4}BC} \lambda^{BC} \\ &= r^{-(n-1)} \sqrt{\det \hat{g}_\Sigma(r)} \eta_{A_1 \dots A_{n-4}BC} \lambda^{BC} \\ &= O(r), \end{aligned} \quad (3.11)$$

as desired. The claim about  $\omega_{AA_1 \dots A_{n-4}}$  is established by a similar calculation.  $\square$

Returning to the proof of Theorem 3.1, suppose first that the Killing vector field  $X$  has no zeros, and that all orbits are of principal type. Let  $S$  be a hypersurface transverse to  $X$ , let  $S(r)$  denote the intersection of  $S$  with the level sets of the function

$r$  of (A.1), we then have

$$\int_S \frac{1}{u^4} |\hat{\omega}|^2 * \xi = \pm \lim_{r \rightarrow 0} \int_{S(r)} \frac{1}{u^2} \xi \wedge \hat{\omega}, \quad (3.12)$$

with the  $\pm 1$  factor as in (3.4). In local coordinates of (A.1) we have  $\xi = g(X, \cdot) = O(r^{-2})$  and  $u^2 \geq cr^{-2}$ , while  $\hat{\omega} = O(1)$  by Lemma 3.2. Further, if we choose  $S$  to be asymptotically orthogonal to  $X$ , then the pull-back of  $\xi$  to  $S$  will be  $o(r^{-2})$ . Thus we obtain

$$\int_S \frac{1}{u^4} |\hat{\omega}|^2 * \xi = \lim_{r \rightarrow 0} o(1) = 0.$$

Now  $*\xi = \alpha |K| d \text{vol}_S$ , where  $\alpha$ , the angle between  $S$  and  $K$ , is of constant sign. We can conclude  $\hat{\omega} = 0$ .

When zeros of  $X$  occur, (3.12) becomes

$$\int_S \frac{1}{u^4} |\hat{\omega}|^2 * \xi = \pm \lim_{r \rightarrow 0} \int_{S(r)} \frac{1}{u^2} \xi \wedge \hat{\omega} \mp \lim_{\epsilon \rightarrow 0} \int_{\{\rho=\epsilon\}} \frac{1}{u^2} \xi \wedge \hat{\omega}. \quad (3.13)$$

Here  $\rho = \sqrt{\sum_{i=1}^{\ell} \rho_i^2}$ , with  $\rho_i$  as in (C.12). To finish the proof, we need to show that the boundary integral corresponding to the zeros of  $X$  vanishes. We use the coordinate system of (C.8): we have  $c^{-1}\rho \leq u \leq c\rho$  by (C.26). Then  $\hat{\omega} = O(\rho)$  by (C.28) and  $\xi = O(\rho)$  by (C.15), hence the integrand in the right-hand side of (3.13) is uniformly bounded as  $\epsilon \rightarrow 0$ . By scaling (or by a direct calculation, using the formulae of Appendix C) one sees then that the integral vanishes at least as fast as  $\epsilon^{2\ell-1}$ , whence the result.  $\square$

The above result is reasonably satisfactory from a general relativistic point of view: in that case the solutions of main interest possess spacelike hypersurfaces transverse to the Killing vector field, which imply topological staticity of the associated Riemannian solution. Nevertheless, it seems of interest to look for other hypotheses which will lead to hypersurface-orthogonality of the Killing vector. In the next section we will obtain some such results under the hypothesis that there exists a *twist potential*  $\tau$ , i.e.,  $\omega = d\tau$ .

## 3.2 Solutions with a twist potential

Our next result assumes that  $\omega$  is exact and that  $X$  has no zeros. The case with zeros will be covered in Theorems 3.4 and 3.6, while the question of exactness of  $\omega$  will be addressed in Theorem 3.7; notations and conventions of Appendix B are used.

In the result that follows we assume that  $(\partial M, \gamma)$  is not conformal to the round sphere. That last case is covered by [5] when  $M$  is spin, and by [1] or [27] (together with [11]) regardless of the existence of a spin structure; in those works, it is shown that  $(M, g)$  is then the hyperbolic space. In our context a simple proof can be given

assuming non-degeneracy, for then every continuous isometry descends to the interior, and the result follows by ODE methods.

**Theorem 3.3.** *Let  $(M, g)$  be Einstein, assume that  $(\partial M, \gamma)$  is not conformal to the round sphere, and suppose that  $\hat{X}(0)$  is hypersurface-orthogonal on  $\partial M$ . Assume further that there exists on  $\mathring{\Sigma}$  a  $(n-4)$ -form  $\tau$  such that*

$$\omega = d\tau. \quad (3.14)$$

*If  $X$  has no zeros, then  $X$  is hypersurface-orthogonal on  $M$ .*

Both the  $(n-4)$ -form  $\tau$  of (3.14), as well as its  $M$ -counterpart  $\hat{\tau} = \pi_\Sigma^* \tau$ , will be referred to as *the twist potentials*.

*Proof.* We use the notation of Appendix B throughout. Let  $r$  be the coordinate of (A.1). By Remark A.3, the function  $r$  passes to the quotient  $\Sigma = M/S^1$ , and by an abuse of notation we shall use the same letter for the resulting function. For  $\rho > 0$  set

$$\Sigma(\rho) = \Sigma \setminus (\{r < \rho\} \cup \{p : d(p, \Sigma_{\text{sing}}) < \rho\}) \subset \mathring{\Sigma}.$$

By (B.20) we have

$$d(u^{-3} *_{g_\Sigma} \omega) = 0. \quad (3.15)$$

Taking the exterior product of this equation with  $\tau$  and integrating over  $\Sigma(\rho)$  one has

$$\begin{aligned} 0 &= (-1)^{(n-4)} \int_{\Sigma(\rho)} \tau \wedge d(u^{-3} *_{g_\Sigma} \omega) \\ &= \int_{\partial \Sigma(\rho)} u^{-3} \tau \wedge *_{g_\Sigma} \omega - \int_{\Sigma(\rho)} u^{-3} d\tau \wedge *_{g_\Sigma} \omega \\ &= \int_{\partial \Sigma(\rho)} u^{-3} \tau \wedge *_{g_\Sigma} \omega - \int_{\Sigma(\rho)} u^{-3} |\omega|_{g_\Sigma}^2 *_{g_\Sigma} 1. \end{aligned} \quad (3.16)$$

The idea is to show that the boundary integral above vanishes when passing with  $\rho$  to zero, yielding  $\omega = 0$ , as desired.

For  $\rho$  small enough  $\partial \Sigma(\rho)$  is a finite union of smooth submanifolds of  $\mathring{\Sigma}$  of co-dimension one. The simplest case is  $\mathring{\Sigma} = \Sigma$ , this occurs when  $X$  has no zeros and all orbits are of principal type, so that  $\Sigma_{\text{sing}} = \emptyset$  and  $\partial \Sigma(\rho)$  equals

$$\partial_\infty \Sigma(\rho) := \{r = \rho\}.$$

In general  $\partial \Sigma(\rho)$  might have further components of the form

$$\partial \Sigma_{\text{sing, iso}}(\rho) := \{p : d(p, \Sigma_{\text{sing, iso}}) = \rho\}$$

and also

$$\partial \Sigma_{\text{sing, 0}}(\rho) := \{p : d(p, \Sigma_{\text{sing, 0}}) = \rho\}.$$

The latter are, however, excluded by our current hypothesis that  $X$  has no zeros on  $M$ .

Lemma 3.2 and the definition (3.14) of  $\tau$  give

$$(n-3)\partial_{[r}\tau_{A_1\dots A_{n-4}]} = \omega_{rA_1\dots A_{n-4}} = O(r), \quad (3.17)$$

where square brackets over a set of indices denote complete anti-symmetrisation with an appropriate combinatorial factor ( $1/((n-3)!)$  in the current case). In dimension four this gives

$$\partial_r \tau = O(r),$$

while in higher dimensions one obtains

$$\partial_{[r}\tau_{A_1\dots A_{n-4}]} = \partial_r \tau_{A_1\dots A_{n-4}} + (-1)^{n-4}\partial_{[A_1}\tau_{A_2\dots A_{n-4}]r} = O(r).$$

By integration we are led to

$$\tau_{A_1\dots A_{n-4}} = \sigma_{A_1\dots A_{n-4}} + O(1), \quad (3.18)$$

where  $\sigma_{A_1\dots A_{n-4}} = 0$  in dimension four and

$$\sigma_{A_1\dots A_{n-4}} := - \int_{r_0}^r (-1)^{n-4} \partial_{[A_1}\tau_{A_2\dots A_{n-4}]r} dr$$

otherwise. Let us use the symbol  $\tilde{d}$  to denote the exterior differential on  $\partial_\infty \Sigma(\rho)$ , at fixed  $\rho$ . Then

$$\begin{aligned} \sigma &:= \frac{1}{(n-4)!} \sigma_{A_1\dots A_{n-4}} dx^{A_1} \wedge \dots \wedge dx^{A_{n-4}} \\ &= \tilde{d} \left[ -\frac{1}{(n-4)!} \left( \int_{r_0}^r (-1)^{n-4} \tau_{A_2\dots A_{n-4}} dr \right) dx^{A_2} \wedge \dots \wedge dx^{A_{n-4}} \right] =: \tilde{d}\tilde{\sigma}. \end{aligned}$$

We note that

$$\int_{\partial_\infty \Sigma(\rho)} u^{-3} \tilde{d}\tilde{\sigma} \wedge *_{g_\Sigma} \omega = \int_{\partial_\infty \Sigma(\rho)} u^{-3} d\tilde{\sigma} \wedge *_{g_\Sigma} \omega = \int_{\partial_\infty \Sigma(\rho)} \tilde{\sigma} \wedge d(u^{-3} *_{g_\Sigma} \omega) = 0,$$

so that Equations (3.10) and (3.18) imply

$$\int_{\partial_\infty \Sigma(\rho)} u^{-3} \tau \wedge *_{g_\Sigma} \omega = \int_{\partial_\infty \Sigma(\rho)} u^{-3} \tau \wedge \lambda = O(\rho^{n-1}),$$

which tends to zero as  $\rho$  tends to zero. If  $\Sigma = \mathring{\Sigma}$  we are done.

Since we have assumed that  $X$  has no zeros, it only remains to analyse the boundary integral around the  $S^1$  orbits with a non-trivial isotropy group. By point 1 of Proposition C.1 below such orbits necessarily form a lower-dimensional subset of  $\Sigma$ , with  $u$  being uniformly bounded in a neighborhood thereof. We are thus integrating a bounded  $(n-2)$ -form over a submanifold, the  $(n-2)$ -area of which shrinks to zero as  $\rho$  tends to zero, which leads to a vanishing contribution in (3.16) in the limit.  $\square$

Next we wish to prove an equivalent of Theorem 3.3 that allows zeros of  $X$ . The proof will again proceed via the identity (3.16), except that we will have now a

supplementary contribution from  $\partial \Sigma_{\text{sing},0}(\rho)$ . Let  $\hat{F}$  be the curvature of the  $U(1)$ -principal bundle of unit normals to  $M_{0,n-2}$ , obtained from the  $U(1)$ -connection  $\gamma_a dx^a$  defined in (C.21); in local coordinates,

$$F := d(\gamma_a dx^a), \quad \hat{F} := \pi_\Sigma^* F. \quad (3.19)$$

We shall use the notation and terminology of Appendix C. We have:

**Theorem 3.4.** *Under the remaining hypotheses of Theorem 3.3 assume instead that  $\bigcup_\ell M_{0,n-2\ell} \neq \emptyset$  and that*

$$\int_{M_{0,n-2}} \hat{\tau} \wedge \hat{F} - \frac{2\pi}{\kappa_1 \kappa_2} \int_{M_{0,n-4}} \hat{\tau} = 0. \quad (3.20)$$

Then

$$\bigcup_{\ell \geq 2} M_{0,n-2\ell} = \emptyset, \quad (3.21)$$

and the conclusions of Theorem 3.3 hold.

*Proof.* By Proposition C.1 the set  $\Sigma_{\text{sing},0}$  is the projection by  $\pi_\Sigma$  of a disjoint union of smooth, non-intersecting, submanifolds of dimension  $n - 2\ell$ ,  $0 \leq \ell \leq n/2$ . It is thus sufficient to consider each such manifold separately. Consider, then, a connected component of  $\partial \Sigma_{\text{sing},0}(\rho)$  which is a projection of a connected component of  $M_{0,n-2\ell}$  for some  $\ell$ , and suppose that

$$\partial \Sigma_{\text{sing},\text{iso}}(\rho) \cap \partial \Sigma_{\text{sing},0}(\rho) = \emptyset$$

for  $\rho$  small enough. (Proposition C.2 shows that this occurs precisely for those components of  $\partial \Sigma_{\text{sing},0}$  for which the associated  $M_{0,n-2\ell}$ 's have all  $\kappa_i$ 's equal to one.)

Consider, first, the case  $\ell = 1$ ; using (C.23), (B.16) and (B.9) we find (recall that  $\hat{\lambda}$  can be identified with  $\lambda$  in the adapted coordinate system used)

$$\lim_{\rho \rightarrow 0} \int_{\partial \Sigma_{\text{sing},0}(\rho)} u^{-3} \tau \wedge *_{g_\Sigma} \omega = \int_{\pi_\Sigma(M_{0,n-2})} \tau \wedge F = \int_{M_{0,n-2}} \hat{\tau} \wedge \hat{F}. \quad (3.22)$$

In dimension  $n$  equal to four the last term in (3.22) is the value of  $\tau$  at the connected component of  $M_{0,n-2}$  under consideration multiplied by the Euler class of the principal  $U(1)$ -bundle of unit vectors normal to  $M_{0,n-2}$ . Regardless of the dimension  $n$ , we have:

**Proposition 3.5.** *When the normal bundle of  $M_{0,n-2}$  is trivial the first integral in (3.20) vanishes.*

*Proof.* If the normal bundle of  $M_{0,n-2}$  is trivial, then  $\gamma_a dx^a$  is defined globally on  $M_{0,n-2}$ , and the integrand in (3.22) integrates out to zero:

$$(-1)^{n-4} \int_{\pi_\Sigma(M_{0,n-2})} \tau \wedge d(\gamma_a dx^a) = \int_{\pi_\Sigma(M_{0,n-2})} (d(\tau \wedge \gamma_a dx^a) - d\tau \wedge \gamma_a dx^a) = 0; \quad (3.23)$$

recall that  $d\hat{\tau} = 0$  on  $M_{0,n-2\ell}$ . (Strictly speaking, for  $\ell \geq 2$  one should do the above calculation on  $\partial\Sigma_{\text{sing},0}(\rho)$  and then pass to the limit, since  $\partial\Sigma_{\text{sing},0}$  does not have a differentiable manifold structure in general for  $\ell \geq 2$  – while  $\tau$  extends by continuity to  $M_{0,n-2\ell}$  in the coordinates of Appendix C, the exterior derivative  $d\tau$  of  $\tau$  might not be defined there).  $\square$

Returning to the proof of Theorem 3.4 suppose, next, that  $\ell \geq 2$ . In the coordinates  $(\rho_1, (\rho_i, \psi_i)_{i=2,\ell-1}, x^a)$  of (C.27) the boundary integrand in (3.16) is of order of  $\rho^{-1}$  while

$$\partial\Sigma_{\text{sing,iso}}(\rho) = \{\rho_1 \geq 0, \rho_1^2 + \dots + \rho_\ell^2 = \rho^2, \psi_i \in [0, 2\pi], x^a \in M_{0,n-2\ell}\}$$

has (coordinate Lebesgue) measure  $O(\rho^{\ell-1})$ , hence

$$\int_{\partial\Sigma_{\text{sing},0}(\rho)} u^{-3} \tau \wedge *_{g_\Sigma} \omega = O(\rho^{\ell-2}). \quad (3.24)$$

The simplest case is then  $\ell \geq 3$ , which immediately gives zero contribution in the limit. Equation (3.24) also shows that the  $M_{0,n-2\ell}$ 's with  $\ell = 2$  give a finite contribution as  $\rho$  tends to zero. Clearly, the only terms that might give a non-zero contribution in the limit are those which arise from the second line of (C.27). If the dimension of  $M$  is four then the second term there does not occur. The first term looks like a total divergence so one is tempted to conclude that it gives a zero contribution when integrated upon. This is, however, deceptive, because the coordinate system used there is singular at the set  $\rho_1 = 0$ , and around each connected component of  $\pi_\Sigma(M_{0,n-4})$  from (C.27) one finds

$$\lim_{\rho \rightarrow 0} \int_{\partial\Sigma_{\text{sing},0}(\rho)} u^{-3} \tau \wedge *_{g_\Sigma} \omega = -\frac{2\pi}{\kappa_1 \kappa_2} \int_{\pi_\Sigma(M_{0,n-4})} \tau = -\frac{2\pi}{\kappa_1 \kappa_2} \int_{M_{0,n-4}} \hat{\tau}; \quad (3.25)$$

the  $1/\kappa_1 = \kappa_1 = \pm 1$  factor arises from a change of orientation. In dimension four each connected component of  $M_{0,n-4}$  is a point, and the integral here is understood as the value of  $\hat{\tau}$  at the point under consideration; (3.25) gives the contribution from the first term in the second line of (C.27) for all  $n$ . It can be checked that for  $n > 4$ , the second term there gives a vanishing contribution in the limit, so that (3.25) holds for all dimensions. (Strictly speaking, at this stage  $\kappa_2 = 1$  in the formula above, as we have assumed that all the nearby orbits have period equal either 0 or  $2\pi$ . However, as shown below, the above formula also gives the boundary contribution around the  $\pi_\Sigma(M_{0,n-4})$ 's in general.)

Clearly (3.25) depends only upon the  $\pi_\Sigma(M_{0,n-4})$ -homology class of the restriction  $\hat{\tau}$  of  $\tau$  to  $\pi_\Sigma(M_{0,n-4})$ .

Let us show how to reduce the general case to the previous one. As explained in Appendix C, in the coordinate patch  $\mathcal{U}_p$  defined there the surface  $\partial \Sigma_{\text{sing,iso}}(\rho)$  takes the form

$$\partial \Sigma_{\text{sing,iso}}(\rho) \cap \mathcal{U}_p = \left\{ \sum_{i=i_2}^{\ell} \rho_i^2 = \rho \right\}.$$

We can deform those surfaces to

$$\left\{ \sum_{i=1}^{\ell} \rho_i^2 = \rho \right\}$$

using the family of surfaces

$$\partial \Sigma(\rho, \delta) := \underbrace{\left\{ \sum_{i=1}^{\ell} \rho_i^2 = \rho, \sum_{i=i_2}^{\ell} \rho_i^2 \geq \delta \right\}}_{\partial \Sigma_1(\rho, \delta)} \cup \underbrace{\left\{ \sum_{i=1}^{\ell} \rho_i^2 \geq \rho, \sum_{i=i_2}^{\ell} \rho_i^2 = \delta \right\}}_{\partial \Sigma_2(\rho, \delta)}$$

with  $0 \leq \delta \leq \rho$ . At fixed  $\rho$ , on  $\partial \Sigma(\rho, \delta)$  the integrand is uniformly bounded while  $\partial \Sigma(\rho, \delta)$  shrinks to a lower-dimensional object as  $\delta$  tends to zero, therefore

$$\lim_{\delta \rightarrow 0} \int_{\partial \Sigma_2(\rho, \delta)} u^{-3} \tau \wedge *_{g_{\Sigma}} \omega = 0.$$

This reduces the problem of calculating the limit, as  $\rho$  goes to zero, of the integral of  $u^{-3} \tau \wedge *_{g_{\Sigma}} \omega$  over  $\partial \Sigma_{\text{sing,iso}}(\rho) \cap \mathcal{U}_p$ , to that of calculating

$$\lim_{\rho \rightarrow 0} \int_{\partial \Sigma_1(\rho, 0)} u^{-3} \tau \wedge *_{g_{\Sigma}} \omega.$$

But this is an integral already considered under the assumption that  $\partial \Sigma_{\text{sing,iso}}$  does not meet  $\partial \Sigma_{\text{sing},0}$  in  $\mathcal{U}_p$ , and an identical analysis applies.

Those components of  $\Sigma_{\text{sing,iso}}$  which do not meet  $\Sigma_{\text{sing},0}$ , or which lie away from the  $\mathcal{U}_p$ 's, are handled as in the proof of Theorem 3.3. Finally, (3.21) is a rephrasing of Proposition C.3.  $\square$

There are various ways to ensure that (3.20) holds: Suppose, for instance, that we are in dimension four, then  $\hat{\tau}$  is a function on  $M$ , defined up to a constant; further  $\hat{\tau}$  is constant on any connected component of  $M_{0,n-2\ell}$ . In this case, when  $\bigcup_{\ell \leq 2} M_{0,n-2\ell}$  is connected, we can choose  $\tau$  to be zero on  $\bigcup_{\ell \leq 2} M_{0,n-2\ell}$ , obtaining a vanishing contribution from  $\bigcup_{\ell \leq 2} M_{0,n-2\ell}$ . Another possibility is to assume that the bundle of unit normals to  $M_{0,n-2}$  is trivial, see Proposition 3.5. If, moreover,  $M_{0,n-4}$  is connected (which will certainly be the case if it is empty), then we can choose again the constant of integration appropriately to achieve the desired equality. One can clearly assume various combinations of the hypotheses above. As a special case, we have obtained:

**Theorem 3.6.** *Under the remaining hypotheses of Theorem 3.3, assume instead that*

$$\left. \begin{array}{l} \bigcup_{\ell \leq 2} M_{0,n-2\ell} \text{ is connected,} \\ \text{the normal bundle to } M_{0,n-2} \text{ is trivial and } M_{0,n-4} \text{ is} \\ \text{connected,} \\ \text{the normal bundle to } M_{0,n-2} \text{ is trivial and } M_{0,n-4} = \emptyset, \\ \bigcup_{\ell \leq 2} M_{0,n-2\ell} = \emptyset, \end{array} \right\} \begin{array}{l} n = 4, \text{ or} \\ n = 4, \text{ or} \\ n \geq 4, \text{ or} \\ n \geq 4. \end{array} \quad (3.26)$$

*Then the conclusions of Theorem 3.4 hold.*

We note that the hypotheses of Theorem 3.6 are stable under perturbations of the metric.

### 3.3 Existence of the twist potential

Let us briefly turn our attention to the question of existence of the twist potential; this will be obviously the case when  $H_{n-3}(M)$  is trivial. Such a hypothesis, however, excludes most situations of interest from a Lorentzian point of view if  $n = 4$ . An alternative possibility is triviality of  $H_{n-3}(\mathring{\Sigma})$  – this covers, in particular, all Lorentzian space-times without black hole regions, with  $M = \Sigma \times S^1$ , and with trivial  $H_{n-3}(\Sigma)$ .

In dimension four, a further family of examples can be obtained as follows: It follows from Lemma 3.2 that, in the coordinate system of (A.1), the one forms

$$\hat{\omega}_A dx^A \quad \text{and} \quad \omega_A dx^A$$

extend by continuity to closed one-forms  $\hat{\omega}_0$  on  $\partial M$ , and  $\omega_0$  on  $\partial \Sigma$ . Clearly a necessary condition for exactness of  $\hat{\omega}$  is exactness of  $\hat{\omega}_0$ . Under some conditions this can be shown to be sufficient:

**Theorem 3.7.** *In dimension  $n = 4$ , suppose that  $X$  has no zeros, then the twist potentials  $\tau$  and  $\hat{\tau}$  exist under either of the following conditions:*

- (1) *There exists a function  $\hat{\tau}_0$  on  $\partial M$  such that  $d\hat{\tau}_0 = \hat{\omega}_0$ , and there are no non-trivial  $L^2$  sections  $\varphi$  of  $\Lambda^1(M)$  which are solutions of the equations*

$$d\varphi = d *_g \varphi = 0. \quad (3.27)$$

- (2) *There are no  $S^1$  orbits with nontrivial isotropy, there exists a function  $\tau_0$  on  $\partial M$  such that  $d\tau_0 = \omega_0$ , and there are no non-trivial  $L^2$  sections  $\varphi$  of  $\Lambda^1(\mathring{\Sigma})$  which are solutions of the equations*

$$d\varphi = d *_g \varphi = 0. \quad (3.28)$$

**Remark 3.8.** Wang [29, Theorem 3.1] gives a condition under which the  $L^2$ -cohomology condition above will hold; in particular it follows from the work of Lee [23] that the



$L^2$ -cohomology condition will be satisfied when the Yamabe invariant of the boundary metric on  $\partial M$  or on  $\partial \Sigma$  is positive.

*Proof.* We first note that existence of  $\tau$  and  $\hat{\tau}$  are equivalent, by projecting down or lifting. In order to prove point 1 consider the equation

$$\nabla_i \nabla^i \tilde{\tau} = \frac{4}{u} \hat{\omega}_i \nabla^i u. \quad (3.29)$$

Lemma 3.2 and the calculations there show that  $u^{-1} \hat{\omega}_i \nabla^i u = O(r^2)$ , so that by point (ii) of Theorem 7.2.1 of [4], together with the Remark (i) following that theorem, there exists a function

$$\tilde{\tau} = \hat{\tau}_0 + O(r^2)$$

which solves (3.29). Equation (B.25) shows that the one-form

$$\varphi := \hat{\omega}_i dx^i - d\tilde{\tau}$$

solves (3.27). Further we have, in the coordinates of Appendix A,

$$\partial_i \rfloor \varphi = O(r) \quad (\text{equivalently } |\varphi|_g^2 = O(r^4)),$$

which implies that  $\varphi \in L^2$ . The vanishing of  $\varphi$  follows from our hypothesis of the vanishing of the first  $L^2$ -cohomology class of  $M$ , hence  $\hat{\omega} = d\tilde{\tau}$ . Point 2 is established in a similar way, using (B.20) instead of (B.25).  $\square$

## A Extensions of conformal isometries from $\partial M$ to $M$

Let  $Y$  be a conformal Killing vector field of  $\partial M$ . Suppose, first, that  $(\partial M, \gamma)$  is a round sphere; as discussed at the beginning of Section 3.2, the pair  $(M, g)$  is then the hyperbolic space. Consider, next, a  $\partial M$  which is *not* a round  $(n-1)$ -dimensional sphere; then the Lelong-Ferrand–Obata theorem shows that we can choose a representative  $\hat{g}(0)$  of the conformal class  $[\gamma]$  so that  $Y$  is a Killing vector thereof. We start with a study of the Killing equation in a neighborhood of  $\partial M$ . It is well known that there exists a defining function  $r$  such that the metric  $g$  takes the form

$$g = r^{-2} \bar{g} = r^{-2} (dr^2 + \hat{g}(r)), \quad \hat{g}(r)(\partial_r, \cdot) = 0. \quad (A.1)$$

on  $[0, \epsilon] \times \partial M$ . Let  $(x^2, \dots, x^n)$  be a local coordinate system on  $\partial M$ . We will work in the coordinate system  $(x^1 = r, x^2, \dots, x^n)$ , and denote by  $r$  the index relative to the first coordinate. We will take upper case Latin letters for the indices relative to the remaining coordinates, and lower case Latin letters for the indices relative to any

component. With that convention, the Christoffel symbols of  $g$  read

$$\Gamma_{rr}^r = -r^{-1}, \quad \Gamma_{rr}^A = \Gamma_{Ar}^r = 0, \quad \Gamma_{AB}^r = r^{-1}\hat{g}_{AB}(r) - \frac{1}{2}\hat{g}'_{AB}(r), \quad (\text{A.2})$$

$$\Gamma_{rA}^C = -r^{-1}\delta_A^C + \frac{1}{2}\hat{g}^{CD}(r)\hat{g}'_{DA}(r), \quad \Gamma_{AB}^C = \hat{\Gamma}_{AB}^C(r). \quad (\text{A.3})$$

Here  $f'$  denotes the derivative of a function  $f$  with respect to  $r$ . The Killing equations,

$$\nabla_i X_j + \nabla_j X_i = 0, \quad (\text{A.4})$$

written out in detail, read

$$\partial_r X_r + r^{-1}X_r = 0, \quad (\text{A.5})$$

$$\partial_r X_A + \partial_A X_r + 2r^{-1}X_A - \hat{g}^{CD}(r)\hat{g}'_{DA}(r)X_C = 0, \quad (\text{A.6})$$

$$\partial_A X_B + \partial_B X_A - 2\hat{\Gamma}_{AB}^C(r)X_C + (\hat{g}'_{AB}(r) - 2r^{-1}\hat{g}_{AB}(r))X_r = 0. \quad (\text{A.7})$$

From (A.5) there exists a function  $\alpha$  such that

$$X_r = \frac{\alpha}{r}, \quad \partial_r \alpha = 0,$$

and, if we define  $\hat{X}_A = r^2 X_A$ , then (A.6) and (A.7) become

$$\partial_r \hat{X}_A + r\partial_A \alpha - \hat{g}^{CD}(r)\hat{g}'_{DA}(r)\hat{X}_C = 0, \quad (\text{A.8})$$

$$\partial_A \hat{X}_B + \partial_B \hat{X}_A - 2\hat{\Gamma}_{AB}^C(r)\hat{X}_C + (r\hat{g}'_{AB}(r) - 2\hat{g}_{AB}(r))\alpha = 0. \quad (\text{A.9})$$

From (A.9),  $\hat{X}(0)$  is a Killing vector field of the boundary if and only if  $\alpha \equiv 0 \Leftrightarrow X_r \equiv 0$ . If that is the case then (A.8) and (A.9) take the form

$$\partial_r \hat{X}_A - \hat{g}^{CD}(r)\hat{g}'_{DA}(r)\hat{X}_C = 0, \quad (\text{A.10})$$

$$\partial_A \hat{X}_B + \partial_B \hat{X}_A - 2\hat{\Gamma}_{AB}^C(r)\hat{X}_C = 0. \quad (\text{A.11})$$

Equation (A.10) has the unique solution

$$\hat{X}_A(r) = \hat{g}_{AC}(r)\hat{X}^C(0), \quad \hat{X}^C(0) := \hat{g}^{CB}(0)\hat{X}_B(0). \quad (\text{A.12})$$

We use now the Taylor development

$$\hat{g}(r) = \hat{g}(0) + O(r^p),$$

where  $p = 1$  in general and  $p = 2$  if  $g$  is Einstein [16]. This yields  $\hat{X}(r) = \hat{X}(0) + O(r^p)$  and  $\hat{\Gamma}(r) = \hat{\Gamma}(0) + O(r^p)$ , thus  $\hat{X}(r)$  given by (A.12) is an approximate solution of (A.11) modulo  $O(r^p)$ . Finally the 1-form

$$X_\infty := 0 dr + r^{-2}(\hat{X}_2(r)dx^2 + \cdots + \hat{X}_n(r)dx^n) \quad (\text{A.13})$$

is an approximate solution of (A.4), with error – in the above coordinates –  $O(r^{p-2})$ .

We wish to show that, under reasonably mild conditions, conformal isometries of  $[\gamma]$  extend to isometries of  $g$ :

**Proposition A.1.** *Let  $(M, g)$  be an asymptotically hyperbolic Einstein manifold and suppose that the operator*

$$\Delta_L + 2(n - 1)$$

*acting on symmetric two-covariant tensors has no  $L^2$  kernel. Then every Killing vector field  $\hat{X}(0)$  on  $\partial M$  extends in a unique way to a Killing vector field  $X$  on  $M$  such that (A.15) holds.*

For  $\hat{X}(0)$ , a (one form associated to a) Killing vector field on  $\partial M$ , consider the boundary value problem

$$\Delta_g X_i = -\text{Ric}(g)_i{}^j X_j, \quad (\text{A.14})$$

$$X - X_\infty \in C_p^{2,\alpha}(M, T^*M). \quad (\text{A.15})$$

We have

**Proposition A.2.** *Let  $(M, g)$  be an asymptotically hyperbolic manifold with  $\text{Ric}(g) < 0$ . Then the problem (A.14)–(A.15) always has a unique solution.*

*Proof.* From Mazzeo [26] (see also [24, Lemma 7.2]) the indicial radius of the Laplace–Beltrami operator  $dd^* + d^*d$  on one-forms, equal to  $\nabla_g^* \nabla_g + \text{Ric}(g)$  on those, is  $(n - 1)/2 - 1$ . Corollary 7.4 in [24] implies then that the indicial radius of the operator

$$P = \nabla_g^* \nabla_g - \text{Ric}(g) \quad (\text{A.16})$$

on one-forms is  $\sqrt{[(n - 1)/2 - 1]^2 + 2(n - 1)} = (n - 1)/2 + 1$ . An integration by parts shows that there are no  $C^2$  compactly supported solutions of (A.14):

$$0 \leq \int -\text{Ric}(g)_{ij} X^i X^j = \int X_i \Delta_g X^i = - \int |\nabla X|^2 \leq 0$$

(recall  $\text{Ric}(g) < 0$ ). Elliptic regularity, completeness of  $M$  together with density results (cf., e.g., [6]) imply that  $P$  has trivial  $L^2$  kernel, and [24, Theorem C] establishes that, in the notations of [24],  $P$  is an isomorphism from  $C_\delta^{k,\alpha}(M; T^1) \equiv C_\delta^{k,\alpha}(M; T^*M)$  to  $C_\delta^{k-2,\alpha}(M; T^*M)$  for all  $\delta$  such that

$$|\delta - (n - 1)/2| < (n - 1)/2 + 1 \iff -1 < \delta < n.$$

(In the case of covector fields, the space  $C_{-1}^{k-2,\alpha}(M; T^*M)$  corresponds to  $O(r^{-2})$  behavior in the coordinates of (A.1).) Let  $\chi$  be a smooth function on  $M$  equal to 1 on  $\{0 \leq r \leq \epsilon/3\}$ , and equal to 0 for  $r \geq 2\epsilon/3$ ; define the 1-form

$$Y = \chi X_\infty,$$

with  $X_\infty$  defined in (A.13). Then  $Y$  is an approximate solution to the Killing equation (A.4) modulo  $O(r^{p-2})$ . (We emphasise that (A.5) and (A.6) are satisfied identically near the boundary, so that the fall-off of the error term is dictated by a possible

error in (A.7).) This implies  $PY = O(r^{p-1}) \in C_p^{k-2,\alpha}(M; T^*M)$  (see, *e.g.*, the proof of [24, Lemma 3.7] for that last property). Thus there exists a unique solution  $Z \in C_p^{k,\alpha}(M; T^*M)$  to the equation

$$PZ = -PY.$$

We set  $X = Y + Z$ ; uniqueness is obvious from what has been said above.  $\square$

*Proof of Proposition A.1.* If we denote by  $B(h) = -\text{tr}_g \nabla h + \frac{1}{2} \nabla \text{tr}_g(h)$ , then the linearisation of the Einstein operator at the Einstein metric  $g$  is [7, Theorem 1.174]

$$D \text{Ein}(g) = \frac{1}{2}(\Delta_L + 2(n-1)) - \text{div}^* B.$$

Let  $X$  be the solution of (A.14)–(A.15). Then  $X$  is in the kernel of the operator  $P$  of (A.16). A two-line calculation shows that

$$B(\mathcal{L}_X g) = P(X) = 0.$$

Now, if  $g$  is an Einstein metric then  $\mathcal{L}_X g$  is in the kernel of  $D \text{Ein}(g)$  whatever the vector field  $X$ : if  $\phi_t$  denotes the (perhaps local) flow of  $X$ , then

$$0 = \frac{d}{dt}(\phi_t^*(\text{Ein}(g)))|_{t=0} = \frac{d}{dt}(\text{Ein}(\phi_t^* g))|_{t=0} = D \text{Ein}(g) \mathcal{L}_X g.$$

It thus follows that

$$(\Delta_L + 2(n-1))\mathcal{L}_X g = 0.$$

Now, Theorem C and Proposition D of [24] show that the operator  $\Delta_L + 2(n-1)$  is an isomorphism from  $C_\delta^{k,\alpha}(M, S_2)$  to  $C_\delta^{k-2,\alpha}(M, S_2)$  for all  $\delta$  such that

$$|\delta - (n-1)/2| < (n-1)/2 \iff 0 < \delta < n-1.$$

Here we use the symbol  $S_2$  to denote the bundle of symmetric two-covariant tensors; in the notation of [24] the space  $C_0^{k,\alpha}(M, S_2)$  corresponds to  $O(r^{-2})$  behavior in the coordinates of (A.1). From what has been said we have  $\mathcal{L}_X g = O(r^{p-2})$  in local coordinates near the boundary, which can be written as  $\mathcal{L}_X g \in C_p^{k,\alpha}(M, S_2)$ . Since  $p > 0$  the isomorphism property gives

$$\mathcal{L}_X g = 0. \quad \square$$

**Remark A.3.** As  $X$  is a Killing vector field,  $\hat{X}$  satisfies (A.10) and (A.11), in particular the field of covectors  $\hat{X}_A(r_0) = \hat{g}_{AC}(r_0)\hat{g}^{CB}(0)\hat{X}_B(0)$  satisfies the Killing equations on the hypersurface  $\{r = r_0\}$ . This is equivalent to the statement that  $X$  is tangent to the level sets of  $r$  with  $\hat{X}^A(r) = \hat{X}^A(0)$ . Further, in the coordinate system of (A.1),

$$\xi := g(X, \cdot) = r^{-2}(\hat{X}_2(r)dx^2 + \cdots + \hat{X}_n(r)dx^n). \quad (\text{A.17})$$

## B The norm and twist equations

Let  $(M, g)$  be an  $n$ -dimensional Riemannian or Lorentzian space-time with a Killing vector field  $X$ ,

$$\nabla_i X_j + \nabla_j X_i = 0. \quad (\text{B.1})$$

It is well known that (B.1) implies the equation

$$\nabla_i \nabla_j X^k = R_{\ell ij}{}^k X^\ell, \quad (\text{B.2})$$

in particular

$$\nabla^j \nabla_j X^k = -\text{Ric}^k{}_j X^j. \quad (\text{B.3})$$

Let us, locally, write the metric in the form (3.1):

$$g = \eta u^2 (d\phi + \theta)^2 + g_\Sigma, \quad \theta(\partial_\phi) = g_\Sigma(\partial_\phi, \cdot) = 0, \quad X = \partial_\phi, \quad (\text{B.4})$$

where  $g_\Sigma$  is the metric induced by  $g$  on the distribution  $X^\perp \subset TM$ , and  $\eta = \pm 1$  according to whether  $X$  is spacelike ( $\eta = 1$ ) or timelike ( $\eta = -1$ ). The metric  $g_\Sigma$  is the natural metric on the orbit space  $\Sigma$  [19]. One can also think of  $\Sigma$  as of any hypersurface transverse to  $X$ , regardless of the structure of the flow of  $X$ ; one should then, however, not confuse  $g_\Sigma$  with the metric induced by  $g$  on  $\Sigma$ . We will be interested in the equations on  $\Sigma$ ; an efficient way of obtaining those is provided by the projection formalism of Geroch [19]. We will be working away from the set of zeros of  $g(X, X)$ . Let

$$P: TM \rightarrow TM$$

denote the orthogonal projection on  $X^\perp$ , we will also use the symbol  $P$  to denote the obvious extension of  $P$  to other tensor bundles. We note that

$$u = \sqrt{\eta g(X, X)} \quad (\text{B.5})$$

(which can be used as the definition of  $u$  regardless of the decomposition (B.4)) and we set

$$n := \frac{X}{u}. \quad (\text{B.6})$$

We then have

$$P(Y) = Y - \eta g(Y, n)n = (\delta_j^i - \eta n^i n_j) Y^j \partial_i.$$

If  $Y$  and  $Z$  are tangent to  $\Sigma$ , and if  $\hat{Y}$  and  $\hat{Z}$  are  $X$ -orthogonal,  $X$ -invariant lifts of  $Y$  and  $Z$  to  $M$ , then the covariant derivative defined as

$$D_Y Z := P(\nabla_{\hat{Y}} \hat{Z})$$

is the Levi-Civita covariant derivative of  $g_\Sigma$  (see [19]). Let

$$\hat{\lambda} := P(u \nabla X),$$

so that

$$\hat{\lambda}_{ij} = u \nabla_i X_j + X_i u_j - X_j u_i, \quad (\text{B.7})$$

where we have written  $u_j$  for  $\nabla_j u$ . The tensor field  $\hat{\lambda}$  is well defined and smooth away from the set of zeros of  $u$  (at which  $u$  might fail to be differentiable). One has  $X^i \nabla_i u = 0$  by (B.1), and one easily checks that  $\hat{\lambda}$  is an anti-symmetric  $X$ -invariant tensor field on  $M$  which annihilates  $X$ , and thus defines a two-form  $\lambda$  on  $\Sigma$  in the usual way. A convenient way of calculating  $\lambda$  in practice is to introduce

$$\beta := u^{-2} \xi. \quad (\text{B.8})$$

With a little work one finds

$$\hat{\lambda} = u^3 d\beta, \quad (\text{B.9})$$

which clearly leads to

$$d(u^{-3} \hat{\lambda}) = 0. \quad (\text{B.10})$$

It can be seen that  $X$  is (locally) hypersurface orthogonal if and only if  $\hat{\lambda}$  vanishes. Indeed, (B.4) shows that the distribution  $X^\perp$  is (locally) integrable if and only if

$$d\phi + \theta = \frac{\eta}{u^2} g(X, \cdot)$$

is closed; that last condition is precisely the equation  $\hat{\lambda} = 0$ .

Let us derive the equations satisfied by  $\hat{\lambda}$  and  $\lambda$ . Using (B.3) we have

$$\nabla_k \hat{\lambda}_{ij} = u_k \nabla_i X_j + u R_{skij} X^s + \nabla_k X_i u_j - \nabla_k X_j u_i + X_i \nabla_k u_j - X_j \nabla_k u_i. \quad (\text{B.11})$$

Applying a projection to both sides of (B.11) one finds

$$D_k \lambda_{ij} = \frac{1}{u} u_k \lambda_{ij} + u P(R_{skij} X^s) + \frac{1}{u} (\lambda_{ki} u_j - \lambda_{kj} u_i). \quad (\text{B.12})$$

Projections commute with anti-symmetrisations, so that the first Bianchi identity implies

$$D_{[k} \lambda_{ij]} = \frac{3}{u} u_{[k} \lambda_{ij]},$$

where square brackets denote complete anti-symmetrisation. Equivalently,

$$d(u^{-3} \lambda) = 0, \quad (\text{B.13})$$

where  $d$  is taken on  $\hat{\Sigma}$ . This does imply (B.10) by pull-back with  $\pi_\Sigma$ , but the implication the other way round does not seem to be evident.

We want to calculate the divergence of  $\lambda$ . In order to do that we need to work out the  $P(R_{skij} X^s)$  term appearing in (B.12); using the fact that  $n$  is proportional to  $X$

we find

$$\begin{aligned}
 P(R_{skij}X^s) &= P((\delta_j^\ell - n_j n^\ell)R_{sk\ell}X^s) \\
 &= P((R_{skij} - n_j n^\ell R_{sk\ell})X^s) \\
 &= P((\delta_i^m - n_i n^m)(R_{skmj} - n_j n^\ell R_{skm\ell})X^s) \\
 &= (R_{skij} - n_i n^m R_{skmj} - n_j n^\ell R_{sk\ell})X^s,
 \end{aligned}$$

where the last equality arises from the fact that all projections have already been carried out; no projection is needed in the  $k$  index since  $n$  is proportional to  $X$ . Upon a contraction over  $k$  and  $i$  in (B.12) the  $P(R_{skij}X^s)$  term will thus give a contribution

$$(-R_{sj} - 0 + n_j n^\ell R_{s\ell})X^s = 0$$

if  $g$  is Einstein. It follows that, for Einstein metrics  $g$ ,

$$D^i \lambda_{ij} = \frac{1}{u}(u^i \lambda_{ij} + 0 + \underbrace{\lambda_i^i}_{0} u_j - \lambda_{ij} u^i) = 0. \quad (\text{B.14})$$

Equivalently,

$$d(*_{g_\Sigma} \lambda) = 0, \quad (\text{B.15})$$

with  $d$  again taken on  $\mathring{\Sigma}$ . We define the *twist  $n - 3$  form  $\omega$  on  $\mathring{\Sigma}$*  by the equation

$$\omega := *_{g_\Sigma} \lambda. \quad (\text{B.16})$$

Equation (B.15) shows that  $\omega$  is closed, while (B.13) is equivalent to

$$d^*(u^{-3}\omega) = 0, \quad (\text{B.17})$$

Let  $\hat{\omega}$  denote the lift of  $\omega$  to  $M$ ,

$$\hat{\omega} = \pi_\Sigma^* \omega,$$

where  $\pi_\Sigma$  is the projection from  $M$  to  $\Sigma$ . Choosing the orientation of  $\Sigma$  appropriately one finds

$$\hat{\omega}_{\alpha_1 \dots \alpha_{n-3}} = \epsilon_{\alpha_1 \dots \alpha_{n-3} \alpha \beta \gamma} X^\alpha \nabla^\beta X^\gamma \iff \hat{\omega} = *_g(\xi \wedge d\xi), \quad (\text{B.18})$$

where

$$\xi = g(X, \cdot)$$

and where  $\epsilon$  is the volume form on  $M$ . Since exterior differentiation commutes with pull-back we have

$$d\hat{\omega} = d(\pi_\Sigma^* \omega) = \pi_\Sigma^*(d\omega) = 0. \quad (\text{B.19})$$

Summarising, on  $\Sigma$  we have

$$d\omega = d(u^{-3} *_g \omega) = 0, \quad (\text{B.20})$$

while on  $M$  it holds that

$$d\hat{\omega} = d(u^{-3}\hat{\lambda}) = 0. \quad (\text{B.21})$$

It is worthwhile mentioning that so far all the equations were manifestly signature-independent.

We note the following equations for  $u$ :

$$\begin{aligned} \nabla_i u &= \frac{\eta}{u} X^j \nabla_i X_j, \\ \nabla^i \nabla_i u &= \frac{\eta}{u} (-\eta g(\nabla u, \nabla u) + \nabla^i X^j \nabla_i X_j - \text{Ric}_{ij} X^i X^j) \\ &= \frac{1}{u} (g(\nabla u, \nabla u) + \eta \frac{\hat{\lambda}^{ij} \hat{\lambda}_{ij}}{u^2} - \eta \text{Ric}_{ij} X^i X^j). \end{aligned} \quad (\text{B.22})$$

The reader is warned that the  $\hat{\lambda}^{ij} \hat{\lambda}_{ij}$  term above can sometimes be negative when  $g$  is Lorentzian and  $X$  is spacelike; similarly  $g(\nabla u, \nabla u)$  can sometimes be negative for Lorentzian metrics.

We refer to [14] for explicit formulae for the curvature tensor of  $g_\Sigma$ .

For completeness let us recall how this formalism works in dimension four: one then sets

$$\omega_i := \epsilon_{ijkl} X^j \nabla^k X^\ell. \quad (\text{B.23})$$

Here, as before,  $\epsilon_{ijkl}$  is the volume form,

$$\epsilon_{ijkl} = 0, \pm \sqrt{|\det g_{mn}|},$$

with  $\epsilon_{ijkl}$  totally antisymmetric, the sign being  $+$  for positive permutations of 1234. The form  $\omega$  from (B.23) is actually the form  $\hat{\omega}$  from (B.18), but we shall not make a distinction between  $\omega$  and  $\hat{\omega}$  anymore. One has  $X^i \omega_i = 0$  by antisymmetry of  $\epsilon_{ijkl}$ . Working away from the set of zeros of  $u$ , with a little work one finds

$$\nabla_i X_j = \frac{2}{u} X_{[j} \nabla_{i]} u + \frac{\sigma \eta}{2u^2} \epsilon_{ijkl} \omega^k X^\ell, \quad (\text{B.24})$$

where  $\sigma = +1$  in the Riemannian case, and  $\sigma = -1$  in the Lorentzian one. The simplest way of performing the algebra involved in this equation, as well as in (B.25) below, is to consider a frame in which  $X = u e_1$ , with  $\omega$  proportional to  $e_2$ . Comparing with (B.7), one recognises the last term above as  $\lambda_{ij}$ . The divergence of  $\hat{\omega}$  can also be computed directly as follows:

$$\begin{aligned} \nabla^i \hat{\omega}_i &= \epsilon_{ijkl} (\nabla^i X^j \nabla^k X^\ell + X^j R_m{}^{ik\ell} X^m) \\ &= \epsilon_{ijkl} \nabla^i X^j \nabla^k X^\ell \\ &= \frac{4\hat{\omega}_i}{u} \nabla^i u. \end{aligned} \quad (\text{B.25})$$



Equivalently,

$$\nabla^i (u^{-4} \hat{\omega}_i) = 0.$$

Equation (B.19) can be rewritten as

$$\nabla_i \hat{\omega}_j - \nabla_j \hat{\omega}_i = D_i \omega_j - D_j \omega_i = 0. \quad (\text{B.26})$$

It follows that

$$\begin{aligned} \nabla^k \nabla_k \hat{\omega}_i &= \nabla^k \nabla_i \hat{\omega}_k = \nabla_i \nabla^k \hat{\omega}_k + \text{Ric}_{ij} \hat{\omega}^j \\ &= 4 \nabla_i \left( \frac{\hat{\omega}_j \nabla^j u}{u} \right) + \text{Ric}_{ij} \hat{\omega}^j. \end{aligned} \quad (\text{B.27})$$

In the four-dimensional case the last line of (B.13) can be rewritten as

$$\nabla^i \nabla_i u = \frac{1}{u} (g(\nabla u, \nabla u) + \frac{1}{2u^2} \sigma g(\hat{\omega}, \hat{\omega}) - \eta \text{Ric}_{ij} X^i X^j). \quad (\text{B.28})$$

## C The structure of the orbit space near fixed points

In order to analyse the contribution to (3.16) arising from the integral over  $\partial \Sigma_{\text{sing},0}(\rho)$ , we need to recall some results about the structure of  $\partial \Sigma_{\text{sing},0}$ . Since  $\nabla_i X_j$  is antisymmetric, for every  $p \in M$  there exists an ON basis of  $T_p M$  in which  $\nabla_i X_j$  is block-diagonal, with  $\ell$  non-zero anti-symmetric two-by-two blocks eventually followed by a block of zeros; such a basis will be referred to as *a basis adapted to  $\nabla X$* . It follows that the dimension of the set

$$\text{Ker}_p \nabla X := \{Y \in T_p M : \nabla_Y X = 0\} \quad (\text{C.1})$$

is necessarily a number of the form  $n - 2\ell$  for some  $0 \leq \ell \leq n/2$ . For such  $\ell$ 's we define

$$M_{0,n-2\ell} := \{p \in M : X(p) = 0, \dim(\text{Ker}_p \nabla X) = n - 2\ell\}. \quad (\text{C.2})$$

For  $p \in M$  let  $\text{Iso}(p)$  denote the isotropy group of  $p$ . We set

$$M_{\text{iso}} := \{p \in M : X(p) \neq 0, \text{Iso}(p) \neq \text{Id}\}. \quad (\text{C.3})$$

For  $p \in M_{\text{iso}}$  let  $\tau_p \in \{2\pi/n\}_{n \in \mathbb{N}^*}$  denote the period of the orbit of  $X$  through  $p$ , set

$$\begin{aligned} \text{Inv}_p &:= \{Y \in T_p M : (\phi_{\tau_p})_* Y = Y\}, \\ M_{\text{iso},\ell} &:= \{p \in M_{\text{iso}} : \dim(\text{Inv}_p) = \ell\}. \end{aligned} \quad (\text{C.4})$$

The following is certainly well known; we give the proof for completeness, because some elements of the argument will be needed in our further analysis:

### Proposition C.1.

(1) *The  $M_{\text{iso},\ell}$ 's are smooth, totally geodesic  $\ell$ -dimensional submanifolds of  $M$ .*

(2) The  $M_{0,n-2\ell}$ 's are smooth, closed, totally geodesic  $(n-2\ell)$ -dimensional submanifolds of  $M$ . In particular

$$M_{0,n-2i} \cap M_{0,n-2j} = \emptyset \quad \text{for } i \neq j.$$

*Proof.* (1) Let  $p \in M_{\text{iso},\ell}$  and let  $\gamma: [0, s_p) \rightarrow M$  be a maximally extended distance-parameterised geodesic such that  $\gamma(0) = p$  and  $\dot{\gamma}(0) = Y$  for some  $Y \in T_p M$ . If  $Y \in \text{Inv}_p$ , then  $\phi_{\tau_p} \circ \gamma: [0, s_p) \rightarrow M$  is again a maximally extended geodesic through  $p$  with tangent vector  $Y$ , which implies  $\phi_{\tau_p} \circ \gamma(s) = \gamma(s)$  for all  $s \in [0, s_p)$ . It follows that the group orbit through  $\gamma(s)$  has period  $\tau_p$  for  $s$  small enough. Clearly if  $Y \notin \text{Inv}_p$  then we will have  $\phi_{\tau_p} \circ \gamma(s) \neq \gamma(s)$  again for  $s$  small enough, and the result follows.

(2) For  $s \in \mathbb{R}$  let  $\phi_s$  denote the action of  $S^1$  on  $M$ , with  $s$  normalised so that  $2\pi$  is the smallest strictly positive number  $s_*$  for which  $\phi_{s_*}$  is the identity on  $M$ . At points at which  $X$  vanishes we have, for any vector field  $Y$ ,

$$\mathcal{L}_X Y = [X, Y] = \nabla_X Y - \nabla_Y X = -\nabla_Y X,$$

so that

$$\frac{d(\phi_{s*} Y)}{ds} = 0 \quad \text{for } Y \in \text{Ker}_p \nabla X. \quad (\text{C.5})$$

Consider any maximally extended affinely parameterised geodesic  $\gamma: I \rightarrow M$  with  $\gamma(0) = p$ , and with the tangent  $\dot{\gamma}(0) \in \text{Ker}_p \nabla X$ . Then  $\phi_s(\gamma)$  is again a maximally extended affinely parameterised geodesic through  $p$ . Further,

$$\frac{d(\phi_{s*} \dot{\gamma}(0))}{ds} = 0 \quad (\text{C.6})$$

by (C.5), which shows that

$$\left. \frac{d(\phi_s \circ \gamma)(t)}{dt} \right|_{t=0} = \dot{\gamma}(0) \quad \text{for all } s.$$

This implies of course that  $\phi_s(\gamma(t)) = \gamma(t)$  for all  $t \in M$  and  $s \in \mathbb{R}$ , so that all points on  $\gamma$  are fixed points of  $\phi_s$ . Hence

$$\exp_p(\text{Ker}_p \nabla X) \subset \bigcup_{\ell} M_{0,n-2\ell}.$$

If we move away from  $p$  in a direction which is not in  $\text{Ker}_p \nabla X$  then  $X$  immediately becomes non-zero, which shows that there exists a neighborhood of  $p$  such that  $\exp_p(\text{Ker}_p \nabla X)$  coincides with  $M_{0,n-2\ell}$  there, and the fact that  $M_{0,n-2\ell}$  is a smooth embedded totally geodesic submanifold follows.

To prove closedness of  $M_{0,n-2\ell}$  consider normal coordinates centred at  $p$ . After performing a rotation if necessary we may suppose that the basis  $\{\partial_i\}$  is adapted to  $\nabla X$  at  $p$ , so that there exist real numbers  $\kappa_i = \kappa_i(p)$  such that at  $p$  we have

$$\frac{1}{2} \nabla_i X_j dx^i \wedge dx^j = \sum_{i=1}^{\ell} \kappa_i dx^{2i-1} \wedge dx^{2i}. \quad (\text{C.7})$$

Closedness of  $M_{0,n-2\ell}$  is clearly equivalent to the statement that the  $|\kappa_i|$ 's are uniformly bounded away from zero on each of the  $M_{0,n-2\ell}$ 's. It is shown below that the  $\kappa_i$ 's are integers, and continuity of the map  $M_{0,n-2\ell} \ni p \rightarrow \{\kappa_i(p)\} \in \mathbb{R}^\ell$  proves the result.  $\square$

In order to continue our analysis of the geometry near fixed points let  $p \in M_{0,n-2\ell}$ , with  $M_{0,n-2\ell}$  as in (C.2), let  $x^a$  denote any local coordinates on  $M_{0,n-2\ell}$  on a coordinate  $M_{0,n-2\ell}$ -neighborhood  $\mathcal{O}_p \subset M_{0,n-2\ell}$  of  $p$ , and for  $q \in \mathcal{O}_p$  let  $x^A$  denote geodesic coordinates on  $\exp_q\{(T_q M_{0,n-2\ell})^\perp\}$ . Passing to a subset of  $\mathcal{O}_p$  if necessary one obtains thus a coordinate system  $(x^i) = (x^A, x^a)$ , with  $A = 1, \dots, 2\ell$ , on an  $M$ -neighborhood  $\mathcal{U}_p \subset M$  of  $p$  diffeomorphic to  $\mathcal{O}_p \times B_{2\ell}(r)$ , where  $B_{2\ell}(r)$  is a ball of radius  $r$  centred at the origin in  $\mathbb{R}^{2\ell}$ . Since  $M_{0,n-2\ell}$  is compact, it can be covered by a finite number of such coordinate systems. This leads to the following local form of the metric

$$g = \sum_{i=1}^{2\ell} (dx^i)^2 + h + \sum_{A,a} O(\rho) dx^A dx^a + \sum_{A,B} O(\rho^2) dx^A dx^B + \sum_{a,b} O(\rho^2) dx^a dx^b, \quad (\text{C.8})$$

with  $h$  the metric induced by  $g$  on  $M_{0,n-2\ell}$ , where  $\rho$  denote the geodesic distance to  $M_{0,n-2\ell}$ . The  $O(\rho^2)$  character of the  $dx^A dx^B$  error terms is standard; the  $O(\rho^2)$  character of the  $dx^a dx^b$  error terms follows from the totally geodesic character of  $M_{0,n-2\ell}$ . We shall need an anti-symmetry property of the derivatives of the  $g_{aA}$ 's, which we now derive: by construction, the coordinate rays  $s \rightarrow sx^A$  are affinely parameterised geodesics. This gives

$$0 = \frac{d^2 x^a}{ds^2} + \Gamma_{ij}^a \frac{dx^i}{ds} \frac{dx^j}{ds} = \Gamma_{BC}^a \frac{dx^A}{ds} \frac{dx^B}{ds}.$$

Since the vector  $dx^A/ds$  can be arbitrarily chosen at  $s = 0$  this implies

$$0 = \Gamma_{BC}^a|_{x^A=0} \iff (g_{aA,B} + g_{AB,A})|_{\{x^C=0\}} = 0, \quad (\text{C.9})$$

where a comma denotes a partial derivative. (Similar arguments may of course be used to justify the  $O(\rho^2)$  character of the remaining error terms in (C.8).)

Exponentiating (C.7) shows that on each space  $(T_q M_{0,n-2\ell})^\perp$  the one-parameter group of diffeomorphisms  $\phi_s$  generated by  $X$  acts as a rotation of angle  $\kappa_i s$  of the planes  $\text{Vect}\{\partial_{2i-1}, \partial_{2i}\}$ ,  $1 \leq i \leq \ell$ . The definition of geodesic coordinates implies that on  $\mathcal{U}_p$  the Killing vector  $X$  equals

$$X = \sum_{i=1}^{\ell} \kappa_i (x^{2i-1} \partial_{2i} - x^{2i} \partial_{2i-1}). \quad (\text{C.10})$$

This equation is exact; there are no error terms, as opposed to e.g. (C.8). Since  $\phi_{2\pi}$  is the identity we have  $\kappa_i \in \mathbb{Z}^*$ , and since almost all orbits have period  $2\pi$  it follows that at least one  $|\kappa_i|$ ,  $|\kappa_1|$  say, equals one. For  $\ell \geq 2$  by renaming and multiplication

by  $-1$  of the coordinates one can arrange to have

$$1 = |\kappa_1| \leq \kappa_i \leq \kappa_{i+1} \leq \kappa_\ell, \quad 2 \leq i \leq \ell - 1, \quad (\text{C.11})$$

and we will always assume that (C.11) holds. We have assumed that an orientation of  $(T_q M_{0,n-2\ell})^\perp$  has been chosen, and the sign in  $\kappa_1$  is chosen so that the coordinates of (C.10) have the correct orientation. Continuity shows that the  $\kappa_i$ 's are constant over each connected component of  $M_{0,n-2\ell}$ .

We shall denote by  $\rho_i$  and  $\varphi_i$  the polar coordinates in the  $(x^{2i-1}, x^{2i})$  planes

$$x^{2i-1} = \rho_i \cos \varphi_i, \quad x^{2i} = \rho_i \sin \varphi_i, \quad (\text{C.12})$$

so that

$$X = \sum_{i=1}^{\ell} \kappa_i \frac{\partial}{\partial \varphi_i}. \quad (\text{C.13})$$

If all the  $\kappa_i$ 's are ones, then all orbits in  $\mathcal{U}_p$  have period  $2\pi$ , in which case

$$M_{\text{iso}} \cap \mathcal{U}_p = \pi_\Sigma^{-1}(\Sigma_{\text{sing,iso}}) \cap \mathcal{U}_p = \emptyset.$$

Otherwise  $\ell \geq 2$  and there exists a smallest  $i_2$  such that  $\kappa_i \geq 2$  for  $i \geq i_2$ . If  $q$  is such that  $\rho_i(q) = 0$  for  $1 \leq i < j$ , and  $\rho_j(q) > 0$ , then the orbit of  $X$  through  $p$  has period  $2\pi/\kappa_j$ . It follows that an orbit through  $q \in \mathcal{U}_p$  has trivial isotropy if and only if

$$\sum_{i=1}^{i_2-1} \rho_i(q) \neq 0.$$

We have shown:

**Proposition C.2.** *We have*

$$M_{0,n-2\ell} \cap \bar{M}_{\text{iso}} \neq \emptyset \iff \text{there exists } i \text{ such that } \kappa_i \geq 2,$$

*in particular*

$$M_{0,n-2} \cap \bar{M}_{\text{iso}} = \emptyset.$$

To proceed further, we need to understand the structure of  $\Sigma$  near  $\pi_\Sigma M_{0,n-2\ell}$ . We first use the polar coordinates (C.12), and then introduce new angular variables  $\varphi, \psi_i$ , parameterising  $\underbrace{S^1 \times \cdots \times S^1}_{n \text{ factors}}$ , defined as

$$\varphi := \varphi_1, \quad \psi_i := \varphi_i - \kappa_1 \kappa_i \varphi_1, \quad i = 2, \dots, n, \quad (\text{C.14})$$

so that, using (C.13),

$$X(\varphi) = 1, \quad X(\psi_i) = X(\rho_i) = 0.$$

It follows that  $X = \partial_\varphi$ , and that  $(\rho_1, (\rho_i, \psi_i)_{i \geq 2})$ , can be used as local coordinates on  $\mathring{\Sigma}$ . There is a usual ‘‘polar coordinates singularity’’ at the sets  $\{\rho_i = 0, u \neq 0\}$  for

$i \geq 2$ . As already pointed out, for  $i$ 's such that  $\kappa_{i+1} > \kappa_i$  the periodicity of the  $\varphi_i$ 's jumps down from  $2\pi/\kappa_i$  to  $2\pi/\kappa_{i+1}$  at the sets  $\{\rho_1 = \dots = \rho_i = 0, u \neq 0\}$ . This leads to an identical jump of the periodicity of the  $\psi_i$ 's, leading to orbifold singularities of increasing complexity at each of those sets. In conclusion, within the domain of the coordinate system  $(\rho_1, (\rho_i, \psi_i)_{i \geq 2})$  the differentiable part  $\mathring{\Sigma}$  of  $\Sigma$  takes the form

$$\{u > 0\}$$

if all the  $\kappa_i$ 's are ones, and

$$\{u > 0\} \setminus \{\rho_1 = \dots = \rho_{i_2} = 0\}$$

otherwise. Further,  $(\rho_1, (\rho_i, \psi_i)_{i \geq 2})$  provide a well behaved coordinate system of polar type on  $\mathring{\Sigma}$  in a neighborhood of  $\pi_\Sigma M_{0,n-2\ell}$ .

Equations (C.8) and (C.10) imply

$$\xi := g(X, \cdot) = \sum_{i=1}^{\ell} \kappa_i (x^{2i-1} dx^{2i} - x^{2i} dx^{2i-1}) + \nu_a dx^a + \sum_i O(\rho^3) dx^i, \quad (\text{C.15})$$

where

$$\nu_a := g_{aA,B}|_{\{x^C=0\}} X^A x^B. \quad (\text{C.16})$$

At this stage it is adequate to enquire about the geometric character of the objects defined so far. Note that the locally defined coordinates  $x^A$  appearing in (C.8) are only determined modulo  $x^a$ -dependent rotations:

$$x^A \rightarrow \bar{x}^A := \omega^A_B(x^a) x^B, \quad (\text{C.17})$$

where, at each  $x^a$ ,  $\omega^A_B$  is an  $2\ell$  by  $2\ell$  orthogonal matrix that preserves all the spaces  $\text{Vect}\{\partial_{2i-1}, \partial_{2i}\}$ . Suppose, thus, that two coordinate systems  $(\bar{x}^A, \bar{x}^a)$  and  $(x^A, x^a)$  are given, with  $\bar{x}^a = x^a$ , and with  $\bar{x}^A$  related to  $x^A$  via (C.17). It is convenient to put bars on  $g_{AB}$ ,  $\nu_a$ , etc., to denote those objects in the barred coordinate system. One easily finds the following transformation law under (C.17):

$$\frac{\partial \bar{g}_{aA}}{\partial \bar{x}^B}|_{\{\bar{x}^C=0\}} \rightarrow \frac{\partial g_{aA}}{\partial x^B}|_{\{x^C=0\}} = \sum_D \omega^D_A \left( \frac{\partial \bar{g}_{aD}}{\partial \bar{x}^E}|_{\{\bar{x}^C=0\}} \omega^E_B + \omega^D_{B,a} \right). \quad (\text{C.18})$$

It follows that

$$\bar{\nu}_a \rightarrow \nu_a = \bar{\nu}_a + \sum_D \omega^D_{B,a} \omega^D_A X^A x^B. \quad (\text{C.19})$$

Now  $\omega$  is a block-diagonal matrix consisting of two-by-two blocks, each of them of the form

$$\begin{bmatrix} \cos(\theta_i(x^a)) & -\sin(\theta_i(x^a)) \\ \sin(\theta_i(x^a)) & \cos(\theta_i(x^a)) \end{bmatrix}.$$

Inserting this into (C.19) one obtains

$$\bar{\nu}_a \rightarrow \nu_a = \bar{\nu}_a + \sum_{i=1}^{\ell} \kappa_i \left( (x^{2i-1})^2 + (x^{2i})^2 \right) \frac{\partial \theta_i}{\partial x^a}. \quad (\text{C.20})$$

So far we have assumed that  $\bar{x}^a = x^a$ ; this last restriction is removed in a straightforward way, leading to a tensorial transformation law of the right-hand side of (C.20) under the transformation

$$(\bar{x}^A, \bar{x}^a) \rightarrow (\tilde{x}^A = \bar{x}^A, \tilde{x}^a = \phi^a(\bar{x}^b)).$$

It should be emphasised that in general we will not be able to achieve  $\omega = \text{id}$  when going from one coordinate patch  $x^a$  to another on  $M_{0,n-2\ell}$ . This implies that  $\nu_a dx^a$  does *not* transform as a one-form when passing from one  $x^a$ -coordinates patch on  $M_{0,n-2\ell}$  to another, except in the case in which the  $x^A$ 's can be globally “synchronised” over  $M_{0,n-2\ell}$  – this occurs if and only if each of the bundles  $\text{Vect}\{\partial_{2i-1}, \partial_{2i}\}$  is trivial.

In order to evaluate the  $\partial \Sigma_{\text{sing,iso}}(\rho)$  integral in (3.16) we need to calculate

$$u^{-3} *_{g_\Sigma} \omega = u^{-3} \lambda,$$

with  $\lambda$  being defined as the  $\Sigma$ -equivalent of the two-form  $\hat{\lambda}$  of (B.7). The simplest way of doing this proceeds via the calculation of the form  $\beta$  of (B.8), cf. (B.9). First consider the case  $\ell = 1$ , set

$$\gamma_a := g_{a1,2}|_{\{x^C=0\}}; \quad (\text{C.21})$$

the anti-symmetry property (C.9) gives

$$\nu_a = \gamma_a \rho_1^2,$$

hence

$$\xi = \kappa_1 \rho_1^2 d\varphi + \gamma_a \rho_1^2 dx^a + \sum_i O(\rho^3) dx^i, \quad (\text{C.22})$$

so that

$$\beta := u^{-2} \xi = \kappa_1 d\varphi + \gamma_a dx^a + O(\rho) d\rho_1 + \sum_a O(\rho) dx^a. \quad (\text{C.23})$$

Equation (C.20) shows that  $\gamma_a dx^a$  is a connection form on the  $U(1)$ -principal bundle of unit vectors normal to  $M_{0,n-2}$ :

$$\bar{\gamma}_a \rightarrow \gamma_a = \bar{\gamma}_a + \frac{\partial \theta_1}{\partial x^a}. \quad (\text{C.24})$$

In particular the curvature two-form

$$F = d\gamma$$

is a well-defined two-form on  $M_{0,n-2}$ .

Let us return to (C.15)–(C.16) for general  $\ell \geq 2$ ; Equation (C.14) gives

$$\begin{aligned} \xi &= \kappa_1 \rho_1^2 d\varphi + \sum_{i=2}^{\ell} \kappa_i \rho_i^2 (d\psi_i + \kappa_1 \kappa_i d\varphi) + v_a dx^a + \sum_i O(\rho^3) dx^i \\ &= u^2 (\kappa_1 d\varphi + \sum_{i=2}^{\ell} \kappa_i u^{-2} \rho_i^2 d\psi_i + u^{-2} v_a dx^a) \\ &\quad + \sum_{i \geq 1} O(\rho) d\rho_i + \sum_{i \geq 2} O(\rho^2) d\psi_i + \sum_a O(\rho) dx^a, \end{aligned} \quad (\text{C.25})$$

with

$$u = \sqrt{\sum_{i=1}^{\ell} \kappa_i^2 \rho_i^2 + O(\rho^3)}. \quad (\text{C.26})$$

Equations (B.8)–(B.9) together with (B.16) and (C.25) immediately lead to

$$\begin{aligned} u^{-3} *_\Sigma \omega &= u^{-3} \lambda \\ &= d\left(\frac{\kappa_i \rho_i^2}{u^2} d\psi_i\right) + d(u^{-2} v_a dx^a) \\ &\quad + \sum_{i,j \geq 1} O(1) d\rho_j \wedge d\rho_i + \sum_{i \geq 2, j \geq 1} O(\rho) d\rho^j \wedge d\psi_i \\ &\quad + \sum_{i,j \geq 2 \geq 1} O(\rho^2) d\psi_j \wedge d\psi_i + \sum_{a,i} O(1) d\rho_i \wedge dx^a \\ &\quad + \sum_{a,j} O(\rho) d\psi_j \wedge dx^a + \sum_{a,b} O(\rho) dx^a \wedge dx^b, \end{aligned} \quad (\text{C.27})$$

which is used in the proof of Theorem 3.4.

We note the following necessary condition for staticity:

**Proposition C.3.** *If  $(M, g)$  is static, then  $M_{0,n-2\ell} = \emptyset$  for  $\ell > 1$ .*

*Proof.* Calculating directly from (C.15) we find

$$d\xi = -2 \sum_{i=1}^{\ell} \kappa_i dx^{2i-1} \wedge dx^{2i} + dv_a \wedge dx^a + \sum_i O(\rho^2) dx^i,$$

so that

$$\begin{aligned} d\xi \wedge \xi &= -2 \sum_{i \neq j=1}^{\ell} \kappa_i \kappa_j dx^{2j-1} \wedge dx^{2j} \wedge (x^{2i-1} dx^{2i} - x^{2i} dx^{2i-1}) \\ &\quad + \sum_{A,B,a} O(\rho^2) dx^A \wedge dx^B \wedge dx^a + \sum_{i,j,k} O(\rho^3) dx^i \wedge dx^j \wedge dx^k, \end{aligned} \quad (\text{C.28})$$

which clearly does never vanish when  $\ell \geq 2$  on a sufficiently small neighborhood of  $M_{0,n-2\ell}$ .  $\square$

## D A family of non-degenerate black hole solutions

### D.1 An injectivity theorem

We start by proving point (a) of Theorem 2.2:

**Theorem D.1.** *Let  $K_{\max}(x)$  and  $K_{\min}(x)$  denote the largest and the smallest sectional curvature of  $g$  at  $x$ . If for all  $x \in M$  it holds that either  $K_{\max}(x) \leq 0$  or  $K_{\min}(x) \geq -2(n-1)/n$ , then the operator  $\Delta_L + 2(n-1)$  has trivial  $L^2$  kernel.*

*Proof.* We use the notations of Lee [24], except that we work in dimension  $n$ , not  $n+1$ . For all  $x \in M$ , let

$$a(x) = \sup\{(\mathring{R}m_x h_x, h_x)/|h_x|^2, h \in S_0^2\}.$$

From [7, Lemma 12.71] we have that

$$a(x) \leq \min\{(n-2)K_{\max}(x) + n-1, -(n-1) - nK_{\min}(x)\},$$

showing that under the current hypotheses we have  $n-1-a(x) \geq 0$ . The proof in [24, p. 67] establishes then that the operator  $\Delta_L + 2(n-1)$  has trivial kernel.  $\square$

### D.2 Sectional curvatures of generalised Kottler metrics

We consider a generalised Kottler metric,

$$g = \frac{1}{V(r)}dr^2 + V(r)d\theta^2 + r^2\hat{g}, \quad (\text{D.1})$$

where  $\hat{g} = \hat{g}_{AB}dx^A dx^B$  does not depend on  $r$  and  $\theta$ . The non-trivial components of the Riemann tensor are

$$\begin{aligned} R_{r\theta r\theta} &= -\frac{1}{2}V'', \\ R_{rArB} &= -\frac{rV'}{2V}\hat{g}_{AB}, \\ R_{\theta A\theta B} &= -\frac{rV'V}{2}\hat{g}_{AB}, \\ R_{ABCD} &= r^2\hat{R}_{ABCD} - r^2V(\hat{g}_{AC}\hat{g}_{BD} - \hat{g}_{AD}\hat{g}_{BC}). \end{aligned}$$



In particular if  $V(r) = c + r^2 - 2mr^{-(n-3)}$ , we obtain

$$\begin{aligned} R_{r\theta r\theta} &= -1 + (n-3)(n-2)mr^{-(n-1)}, \\ R_{rArB} &= -[(r^2 + m(n-3)r^{-(n-3)})/V]\hat{g}_{AB}, \\ R_{\theta A\theta B} &= -(r^2 + m(n-3)r^{-(n-3)})V\hat{g}_{AB}, \\ R_{ABCD} &= r^2\hat{R}_{ABCD} - r^2V(\hat{g}_{AC}\hat{g}_{BD} - \hat{g}_{AD}\hat{g}_{BC}). \end{aligned}$$

Let  $U = (U^r, U^\theta, U^A) = (U^r, U^\theta, \hat{U})$  and  $W = (W^r, W^\theta, W^A) = (W^r, W^\theta, \hat{W})$  be two orthogonal vectors with norm 1, the sectional curvature of  $\text{span}(U, W)$  is

$$\begin{aligned} K(U, W) &= -\frac{1}{2}V''(r)[U^r W^\theta - W^r U^\theta]^2 \\ &\quad - \frac{rV'(r)}{2} \left\{ \frac{1}{V(r)} \|U^r \hat{W} - W^r \hat{U}\|_{\hat{g}}^2 + V(r) \|U^\theta \hat{W} - W^\theta \hat{U}\|_{\hat{g}}^2 \right\} \\ &\quad + r^2 \hat{R}_{ABCD} U^A W^B U^C W^D - r^2 V(r) (\|\hat{U}\|_{\hat{g}}^2 \|\hat{W}\|_{\hat{g}}^2 - \langle \hat{U}, \hat{W} \rangle_{\hat{g}}^2) \\ &= -\frac{1}{2}V''(r)[U^r W^\theta - W^r U^\theta]^2 \\ &\quad - \frac{rV'(r)}{2} \left\{ \frac{1}{V(r)} \|U^r \hat{W} - W^r \hat{U}\|_{\hat{g}}^2 + V(r) \|U^\theta \hat{W} - W^\theta \hat{U}\|_{\hat{g}}^2 \right\} \\ &\quad + r^2 (\hat{K}(\hat{U}, \hat{W}) - V(r)) (\|\hat{U}\|_{\hat{g}}^2 \|\hat{W}\|_{\hat{g}}^2 - \langle \hat{U}, \hat{W} \rangle_{\hat{g}}^2). \end{aligned}$$

Set

$$(a, b, c) = (V^{-1/2}(r)U^r, V^{1/2}(r)U^\theta, r\hat{U})$$

and

$$(x, y, z) = (V^{-1/2}(r)W^r, V^{1/2}(r)W^\theta, r\hat{W}),$$

so that  $a^2 + b^2 + |c|^2 = x^2 + y^2 + |z|^2 = 1$  and  $ax + by + \langle c, z \rangle = 0$ , where the norm  $|\cdot|$  and the scalar product  $\langle \cdot, \cdot \rangle$  are taken with respect to  $\hat{g}$ . We can rewrite the sectional curvature as

$$\begin{aligned} K(U, W) &= -\frac{1}{2}V''(r)[ay - bx]^2 - \frac{r^{-1}V'(r)}{2} \{ |az - xc|^2 + |bz - yc|^2 \} \\ &\quad + r^{-2}[\hat{K}(\hat{U}, \hat{W}) - V(r)](|c|^2|z|^2 - \langle c, z \rangle^2). \end{aligned}$$

Setting

$$k = \min(-\frac{1}{2}V''(r), -\frac{r^{-1}V'(r)}{2}, r^{-2}[\hat{K}(\hat{U}, \hat{W}) - V(r)])$$

we obtain

$$K(U, W) \geq k([ay - bx]^2 + |az - xc|^2 + |bz - yc|^2 + |c|^2|z|^2 - \langle c, z \rangle^2) = k.$$

Similarly,

$$K(U, V) \leq K := \max(-\frac{1}{2}V''(r), -\frac{r^{-1}V'(r)}{2}, r^{-2}[\hat{K}(\hat{U}, \hat{W}) - V(r)]).$$

Coming back to  $V(r) = c + r^2 - 2mr^{-(n-3)}$ , we further assume that  $\hat{K}(\hat{U}, \hat{W}) = c$  and  $n \geq 4$ . One then finds

$$k = -1 + r^{-(n-1)} \min\{(n-3)(n-2)m, -(n-3)m, 2m\},$$

so that if  $m \geq 0$  then

$$k = -1 - m(n-3)r^{-(n-1)}, \quad K = -1 + m(n-3)(n-2)r^{-(n-1)}. \quad (\text{D.2})$$

while for  $m \leq 0$  one has

$$k = -1 + m(n-3)(n-2)r^{-(n-1)} = -1 - |m|(n-3)(n-2)r^{-(n-1)}, \quad (\text{D.3})$$

$$K = -1 + |m|(n-3)r^{-(n-1)}. \quad (\text{D.4})$$

**A)** If  $m \geq 0$ , we have  $k \geq -2(n-1)/n$  if and only if

$$mr_+^{-(n-1)} \leq \frac{n-2}{n(n-3)}, \quad (\text{D.5})$$

where  $r_+$  is the unique positive solution of

$$V(r_+) = 0 \iff mr_+^{-(n-3)} = \frac{1}{2}(c + r_+^2). \quad (\text{D.6})$$

On the other hand,  $K \leq 0$  will hold if and only if

$$mr_+^{-(n-1)} \leq \frac{1}{(n-3)(n-2)}. \quad (\text{D.7})$$

Since the right-hand side of (D.5) is larger than that of (D.7) for  $n > 4$ , with equality for  $n = 4$ , the former condition is less restrictive than the latter. For further use we note that

$$mr_+^{-(n-1)} = r_+^{-2}mr_+^{-(n-3)} = \frac{1}{2}\left(1 + \frac{c}{r_+^2}\right). \quad (\text{D.8})$$

**a)** For  $c = 1$  the left-hand side of (D.5) is strictly larger than  $1/2$  for  $m > 0$  by (D.8), while the right-hand side is less than or equal to  $1/2$  when  $n \geq 4$ , and our non-degeneracy criterion in terms of  $k$  does not apply. Similarly one finds that some sectional curvatures are always positive at  $r = r_+$ .

If we assume that the sectional curvatures of  $\hat{g}$  are equal to  $c = 1$ , and that the manifold  $N^{n-2}$  carrying the metric  $\hat{g}$  is compact, then  $(N^{n-2}, \hat{g})$  is clearly of positive Yamabe type, and one can likewise attempt to use point (b) of Theorem 2.2 to prove non-degeneracy. Unfortunately, it turns out that the sectional curvature inequality there is always violated at  $r_+$ .

**b)** If  $c = 0$  then  $r_+ = (2m)^{1/(n-1)}$ , giving  $1/2 = 1/2$  for  $n = 4$  in (D.5), without restrictions on  $m$ . However, for  $n \geq 5$  the right-hand side of (D.5) is always smaller than one half. Similarly the inequality of point (b) of Theorem 2.2 always fails.

**c)** If  $c = -1$  then the map

$$[1, \infty) \ni r_+(m) \longleftrightarrow m(r_+) \in [0, \infty)$$

is a bijection, and for all  $0 \leq m \leq m_+$  from (D.5) we obtain non-degeneracy, where  $m_+ = \infty$  if  $n = 4$ . For  $n \geq 5$  the value of  $m_+$  can be found by first solving (D.5) in terms of  $r_+$  using (D.8),

$$r_+(m_+) = \sqrt{\frac{n(n-3)}{n(n-3) - 2(n-2)}} = \sqrt{\frac{n(n-3)}{(n-1)(n-4)}}.$$

Equation (D.6) can then be used to calculate  $m_+ = m_+(n)$ :

$$m_+(n) = \begin{cases} \infty, & n = 4; \\ \frac{(n-2)}{(n-1)(n-4)} \left( \frac{n(n-3)}{(n-1)(n-4)} \right)^{\frac{n-3}{2}}, & n \geq 5. \end{cases} \quad (\text{D.9})$$

**B)** If  $m < 0$ , we have  $k \geq -2(n-1)/n$  if and only if

$$|m|r_+^{-(n-1)} \leq \frac{1}{n(n-3)}, \quad (\text{D.10})$$

while  $K \leq 0$  is equivalent to

$$|m|r_+^{-(n-1)} \leq \frac{1}{(n-3)}, \quad (\text{D.11})$$

this last condition being less restrictive than (D.10).

The only case of interest is  $c = -1$ , as  $V$  has no zeros otherwise. The map

$$\left[ r_{\min} := \sqrt{\frac{n-3}{n-1}}, 1 \right] \ni r_+(m) \longleftrightarrow m(r_+) \in \left[ \frac{1}{2}(r_{\min}^{n-1} - r_{\min}^{n-3}), 0 \right]$$

is a bijection, and for all  $m_- \leq m < 0$  from (D.11) we obtain negative sectional curvatures, where

$$r_+(m_-) = r_{\min} = \sqrt{\frac{n-3}{n-1}}, \quad (\text{D.12})$$

$$m_- = m_-(n) = -\frac{1}{n-1} \left( \frac{n-3}{n-1} \right)^{\frac{n-3}{2}}. \quad (\text{D.13})$$

Recall that  $r_{\min}$  given by (D.12) corresponds to the smallest value of  $r_+(m)$  for which a regular solution exists. Equations (D.12)–(D.13) show that non-degeneracy holds in the whole range of negative masses compatible with a singularity-free metric. Summarising, we have proved:

**Proposition D.2.** *Let  $V(r) = -1 + r^2 - 2mr^{-(n-3)}$ , suppose that  $\hat{g}$  is a metric of constant sectional curvature equal to  $-1$  on a compact manifold  $N^{n-2}$ , then for  $n \geq 4$  and for<sup>7</sup>  $m \in (m_-(n), m_+(n)]$ , as given by (D.13) and (D.9), the metric (D.1) is non-degenerate. In dimension four all singularity-free such solutions are non-degenerate.*

## References

- [1] M. T. Anderson, Einstein metrics with prescribed conformal infinity on 4-manifolds, 2001, math.DG/0105243.
- [2] —, Boundary regularity, uniqueness and non-uniqueness for AH Einstein metrics on 4-manifolds, *Adv. Math.* 179 (2003), 205–249, math.DG/0104171.
- [3] M. T. Anderson, P. T. Chruściel, and E. Delay, Non-trivial, static, geodesically complete vacuum space-times with a negative cosmological constant, *J. High Energy Phys.* 10 (2002), 063, gr-qc/0211006.
- [4] L. Andersson and P. T. Chruściel, On asymptotic behavior of solutions of the constraint equations in general relativity with “hyperboloidal boundary conditions”, *Dissertationes Math.* 355 (1996), 1–100.
- [5] L. Andersson and M. Dahl, Scalar curvature rigidity for asymptotically locally hyperbolic manifolds, *Ann. Global Anal. Geom.* 16 (1998), 1–27, dg-ga/9707017.
- [6] T. Aubin, Espaces de Sobolev sur les variétés Riemanniennes, *Bull. Sci. Math.* (2) 100 (1976), 149–173.
- [7] A. L. Besse, *Einstein manifolds*, *Ergeb. Math. Grenzgeb.* (3) 10, Springer-Verlag, Berlin, New York, Heidelberg 1987.
- [8] O. Biquard, Métriques d’Einstein asymptotiquement symétriques (Asymptotically symmetric Einstein metrics), *Astérisque* 265 (2000), vi+109 pp.
- [9] J. Bjoraker and Y. Hosotani, Monopoles, dyons and black holes in the four-dimensional Einstein–Yang–Mills theory, *Phys. Rev. D* 62 (2000), 043513.
- [10] G. E. Bredon, *Introduction to compact transformation groups*, *Pure Appl. Math.* 46, Academic Press, New York 1972.
- [11] U. Christ and J. Lohkamp, in preparation, 2003.
- [12] P. T. Chruściel, E. Delay, J. M. Lee, and D. N. Skinner, Boundary regularity of conformally compact Einstein metrics, math.DG/0401386, 2003.
- [13] P. T. Chruściel and W. Simon, Towards the classification of static vacuum spacetimes with negative cosmological constant, *J. Math. Phys.* 42 (2001), 1779–1817, gr-qc/0004032.
- [14] R. Coquereaux and A. Jadczyk, *Riemannian geometry, fiber bundles, Kaluza–Klein theories and all that . . .* World Sci. Lecture Notes Phys. 16, World Scientific Publishing Co., Singapore 1985.

---

<sup>7</sup>The case  $m = m_-(n)$  corresponds to  $V'(r_{\min}) = 0$ , which leads to a cylindrical end for the metric (D.1), and is therefore excluded by the requirement that the associated Riemannian manifold be conformally compact. The corresponding Lorentzian solution is a regular black hole with vanishing surface gravity.

- [15] E. Delay, Essential spectrum of the Lichnerowicz Laplacian on 2-tensors on asymptotically hyperbolic manifolds, *J. Geom. Phys.* 43 (2002), 33–44.
- [16] C. Fefferman and C. R. Graham, Conformal invariants, in: *Élie Cartan et les mathématiques d'aujourd'hui* (Lyon 1984), *Astérisque* (1985), Numéro hors-série, 95–116.
- [17] R. Fintushel, Circle actions on simply connected 4-manifolds, *Trans. Amer. Math. Soc.* 230 (1977), 147–171.
- [18] —, Classification of circle actions on 4-manifolds, *Trans. Amer. Math. Soc.* 242 (1978), 377–390.
- [19] R. Geroch, A method for generating solutions of Einstein's equations, *J. Math. Phys.* 12 (1971), 918–924.
- [20] G. W. Gibbons and S. W. Hawking, Classification of gravitational instanton symmetries, *Comm. Math. Phys.* 66 (1979), 291–310.
- [21] C. R. Graham and J. M. Lee, Einstein metrics with prescribed conformal infinity on the ball, *Adv. Math.* 87 (1991), 186–225.
- [22] S. de Haro, S. N. Solodukhin, and K. Skenderis, Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence, *Comm. Math. Phys.* 217 (2001), 595–622, hep-th/0002230.
- [23] J. M. Lee, The spectrum of an asymptotically hyperbolic Einstein manifold, *Comm. Anal. Geom.* 3 (1995), 253–271.
- [24] —, Fredholm operators and Einstein metrics on conformally compact manifolds, math.DG/0105046, 2001, Mem. Amer. Math. Soc., in press.
- [25] M. Mars and W. Simon, A proof of uniqueness of the Taub–Bolt instanton, *J. Geom. Phys.* 32 (1999), 211–226.
- [26] R. Mazzeo, The Hodge cohomology of a conformally compact metric, *J. Differential Geom.* 28 (1988), 309–339.
- [27] J. Qing, On the uniqueness of AdS space-time in higher dimensions, *Ann. Henri Poincaré* 5 (2004), 245–260, math.DG/0310281.
- [28] W. Simon, NUTs have no hair, *Classical Quantum Gravity* 12 (1995), L125–L130.
- [29] X. Wang, On conformally compact Einstein manifolds, *Math. Res. Lett.* 8 (2001), 671–688.
- [30] E. Winstanley and O. Sarbach, On the linear stability of solitons and hairy black holes with a negative cosmological constant: The even-parity sector, *Classical Quantum Gravity* 19 (2002), 689–724, gr-qc/0111039.

# The conformal boundary of anti-de Sitter space-times

Charles Frances

Laboratoire de Topologie et Dynamique, Université Paris-Sud  
Bâtiment 430, 91405 Orsay, France  
email: `Charles.Frances@math.u-psud.fr`

## 1 Introduction

Given a non compact space-time  $(M, g)$  (i.e. a manifold  $M$  endowed with a Lorentzian metric  $g$ ), the study of the geometrical asymptotic properties of  $M$  is of great mathematical as well as physical interest. This kind of study was pioneered by R. Penrose (see [P]) by associating a conformal boundary to some space-times. More precisely, a *conformal boundary completion* of a space-time  $(M, g)$  is a manifold  $\bar{M}$  with boundary  $\partial M$  endowed with a conformal class of Lorentzian metrics  $[\bar{g}]$  satisfying the following conditions:

- (i) The interior of  $(\bar{M}, [\bar{g}])$  is conformally diffeomorphic to  $(M, g)$ .
- (ii) For a metric  $\bar{g}$  in the conformal class  $[\bar{g}]$  there is a smooth function  $\rho$  on  $\bar{M}$  such that  $\rho^{-1}(\{0\}) = \partial M$ ,  $d\rho \neq 0$  on  $\partial M$ , and  $\bar{g} = \rho^2 g$  on  $M$ .

Notice that if  $\bar{g} \in [\bar{g}]$ , the causal character of  $\bar{g}|_{\partial M}$  can be non-constant on  $\partial M$ . For example, if  $B$  is an open (Euclidian) ball in Minkowski space, the Minkowski metric restricted to  $\partial B$  is Lorentzian at some points, Riemannian or degenerate at others.

In [P], Penrose constructed a conformal boundary completion for several families of space-times, but in the general case two natural questions remain widely open: under which minimal hypotheses does such a conformal boundary completion exist? When it exists, do we have some kind of uniqueness result?

In the Riemannian context a lot of works concerning *conformally compact Einstein metrics* were done (see [A], [C-H], [Bi] and many others), and some partial results are known about the two previous questions.

The case of compact conformal boundary completions (or conformal compactifications) is not the only relevant. In fact, very basic and natural space-times involve non compact conformal boundary completions. One example which will be fundamental in what follows is anti-de Sitter space  $\widetilde{\text{AdS}}_{n+1}$  (the simply connected complete Lorentzian manifold of curvature  $-1$ ). This space admits as conformal boundary *Einstein's static universe*  $\widetilde{\text{Ein}}_n$  (see Section 2), which is topologically  $\mathbb{R} \times \mathbb{S}^{n-1}$ . Building

a conformal boundary completion of other anti-de Sitter space-times (i.e. lorentzian manifolds of constant curvature  $-1$ ) can be also useful, as shows the example of BTZ blackholes (see [BTZ1],[BTZ2]). These blackholes are particular cases of *Kleinian anti-de Sitter structures*, namely the quotient of some open subset  $\Omega \subset \widetilde{\text{AdS}}_{n+1}$  by a discrete subgroup of isometries  $\Gamma \subset \widetilde{\text{O}(2, n)}$ . In this paper we will focus on a very special kind of Kleinian structures, namely the *complete* anti-de Sitter structures. In other words, we will deal mainly with space-times of the form  $\widetilde{\text{AdS}}_{n+1}/\Gamma$ , for a discrete  $\Gamma \subset \widetilde{\text{O}(2, n)}$  acting properly discontinuously on  $\widetilde{\text{AdS}}_{n+1}$ . The aim of this article is to build a conformal boundary completion for these complete structures. We will also get a uniqueness result (Theorem 1) and prove that in this context the conformal structure on the boundary  $(\partial M, [\bar{g}]|_{\partial M})$  determines completely  $(M, g)$  (Theorem 2). In Section 5, we give an illustration of these constructions on explicit examples.

Notice that the restriction to the case of complete structures is not “physically realistic”. Indeed, to get space-times without causality pathologies (such as closed causal geodesics) it is in general necessary to consider only quotients of *strict* open subsets of  $\widetilde{\text{AdS}}_{n+1}$  (as it is the case for BTZ blackholes). Nevertheless, we think that the methods presented here will be useful to deal with the general case of Kleinian structures.

## 2 Anti-de Sitter space and Einstein’s universe

We recall here some basic properties of Einstein’s universe and anti-de Sitter space. We refer to [HE], [O’N], [S] and [Fr1] for more details.

### 2.1 Einstein’s static universe

Let  $\mathbb{R}^{2,n+1}$  be the space  $\mathbb{R}^{n+3}$ , which is endowed with the quadratic form  $q^{2,n+1}(x) = -2x_1x_{n+3} - 2x_2x_{n+2} + x_3^2 + \cdots + x_n^2 + x_{n+1}^2$ . We call  $C^{2,n+1}$  the isotropic cone of  $q^{2,n+1}$ . The projection of  $C^{2,n+1}$  on  $\mathbb{R}P^{n+2}$  is a smooth quadric endowed with a natural Lorentzian conformal structure inherited from  $\mathbb{R}^{2,n+1}$ . This quadric together with its natural structure is called the *compact Einstein’s universe* of dimension  $n + 1$ , denoted by  $\text{Ein}_{n+1}$ . The group  $\text{PO}(2, n + 1)$  acts conformally on  $\text{Ein}_{n+1}$  and this is in fact exactly the conformal group of  $\text{Ein}_{n+1}$ . The space  $\text{Ein}_{n+1}$  is not simply connected and admits  $\mathbb{R} \times \mathbb{S}^n$  as universal cover. The lift of the conformal class of  $\text{Ein}_{n+1}$  is the conformal class of the metric  $-dt^2 + g_{\mathbb{S}^n}$ . The pair  $(\mathbb{R} \times \mathbb{S}^n, [-dt^2 + g_{\mathbb{S}^n}])$  is called *static Einstein’s universe* and is denoted by  $\widetilde{\text{Ein}}_{n+1}$ . Any conformal transformation of  $\text{Ein}_n$  can be lifted into a conformal transformation of  $\widetilde{\text{Ein}}_n$ , and we denote by  $\widetilde{\text{O}(2, n + 1)}$  the lifting of the whole Lorentzian Möbius group  $\text{O}(2, n + 1)$ . The identity component  $\text{SO}^o(2, n + 1)$  has a center isomorphic to  $\mathbb{Z}$  and generated by  $\zeta : (t, x) \mapsto (t + \pi, -x)$ . The space  $\text{Ein}_{n+1}$  is just the quotient of  $\widetilde{\text{Ein}}_{n+1}$  by the action of the center.

## 2.2 Lightlike geodesics

It is well known that in pseudo-Riemannian conformal geometry lightlike geodesics have a conformal meaning. Indeed, all the metrics of a same conformal class have the same lightlike geodesics (if one forgets the parametrization). On  $\widetilde{\text{Ein}}_{n+1}$ , lightlike geodesics admit a parametrization of the form:  $t \mapsto (t, c(t))$ , where  $t \mapsto c(t)$  is a geodesic of  $\mathbb{S}^n$  (endowed with its canonical Riemannian metric of curvature  $+1$ ). Any lightlike geodesic is left invariant by the action of the center (so that lightlike geodesics on  $\text{Ein}_{n+1}$  are circles).

Given a point  $p$  in  $\widetilde{\text{Ein}}_{n+1}$ , the *lightcone with vertex  $p$* , denoted by  $C(p)$ , is the set of lightlike geodesics passing through  $p$ . Lightcones are not smooth submanifolds of  $\widetilde{\text{Ein}}_{n+1}$ . The singular points are exactly the points  $\zeta^k.p$ , for  $k \in \mathbb{Z}$ . Removing its singular points to a lightcone, one gets a countably infinite family of connected components, each one diffeomorphic to the product  $\mathbb{R} \times \mathbb{S}^{n-1}$ .

## 2.3 The conformal boundary completion of $\widetilde{\text{AdS}}_{n+1}$

In  $\mathbb{R}^{2,n+1}$  we denote by  $\mathbb{R}^{2,n}$  the subspace generated by  $e_1, \dots, e_n, e_{n+2}, e_{n+3}$ . The projection of  $\mathbb{R}^{2,n} \cap C^{2,n+1}$  on  $\text{Ein}_{n+1}$  is a sub-Einstein's universe of codimension 1 that we call  $\text{Ein}_n$ . The complementary of  $\text{Ein}_n$  in  $\text{Ein}_{n+1}$  is exactly the projection of the set  $Q = \{u + e_{n+1} \in \mathbb{R}^{2,n+1} \mid u \in \mathbb{R}^{2,n}, q^{2,n+1}(u) = -1\}$ . It is, conformally, the space  $\text{AdS}_{n+1}$  (namely the quotient of the anti-de Sitter space  $\widetilde{\text{AdS}}_{n+1}$  by the center of its isometry group). The subspace  $\text{Ein}_n$  lifts into a subspace  $\widetilde{\text{Ein}}_n \subset \widetilde{\text{Ein}}_{n+1}$ . In this model the complementary of  $\widetilde{\text{Ein}}_n$  in  $\widetilde{\text{Ein}}_{n+1}$  has two connected components, each one conformally equivalent to  $\widetilde{\text{AdS}}_{n+1}$ . We chose one of these components that we call  $\widetilde{\text{AdS}}_{n+1}$ . The union  $\widetilde{\text{AdS}}_{n+1} \cup \widetilde{\text{Ein}}_n = \widetilde{\text{AdS}}_{n+1}$  is a submanifold of  $\widetilde{\text{Ein}}_{n+1}$ , with boundary  $\widetilde{\text{Ein}}_n$  and interior  $\widetilde{\text{AdS}}_{n+1}$ . The canonical conformal structure on  $\widetilde{\text{Ein}}_{n+1}$  induces a canonical conformal structure  $[\bar{g}_{\text{can}}]$  on  $\widetilde{\text{AdS}}_{n+1}$ , and  $(\widetilde{\text{AdS}}_{n+1}, [\bar{g}_{\text{can}}])$  is a conformal boundary completion of  $\widetilde{\text{AdS}}_{n+1}$ .

The subgroup  $\widetilde{\text{O}}(2, n) \subset \widetilde{\text{O}}(2, n+1)$  which leaves  $\mathbb{R}^{2,n} \subset \mathbb{R}^{2,n+1}$  invariant, acts isometrically on  $\widetilde{\text{AdS}}_{n+1}$ , and conformally on  $\widetilde{\text{AdS}}_{n+1}$ .

## 3 Conformal dynamics

### 3.1 Cartan's decomposition

The group  $\widetilde{\text{O}}(2, n)$  is not a matrix group, and it is not so easy to understand its dynamics on  $\widetilde{\text{Ein}}_n$ , or  $\widetilde{\text{AdS}}_{n+1}$ . Nevertheless, some specific subgroups or subsets of  $\widetilde{\text{O}}(2, n)$  are simple to describe, and help in the understanding of the whole group:



- The center of  $\widetilde{\mathrm{SO}^o(2, n)}$ , isomorphic to  $\mathbb{Z}$ , and generated by the transformation  $\zeta: (t, x) \mapsto (t + \pi, -x)$  (we denote by  $(t, x)$  the points of Einstein's universe  $\mathbb{R} \times \mathbb{S}^{n-1}$ ).
- The subset  $K$  consisting of transformations of the form:  $(t, x) \mapsto (t + a, \sigma.x)$ , with  $a \in [0, \pi[$  and  $\sigma \in \mathrm{O}(n - 2)$ . Note that  $K$  is relatively compact.
- The abelian subgroup  $A^+$ , which projects injectively in  $\mathrm{O}(2, n)$  onto the subgroup of matrices

$$\begin{pmatrix} e^\lambda & & & & & \\ & e^\mu & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & e^{-\mu} \\ & & & & & & e^{-\lambda} \end{pmatrix}$$

with  $\lambda \geq \mu \geq 0$ .

Now every element  $g$  of  $\widetilde{\mathrm{O}(2, n)}$  can be written  $g = \zeta^{l(g)} k_1(g) a^+(g) k_2(g)$ , with  $l(g) \in \mathbb{Z}$ ,  $k_1(g)$  and  $k_2(g)$  in  $K$ , and  $a^+(g) \in A^+$ . This decomposition is called *Cartan's decomposition* of  $\widetilde{\mathrm{O}(2, n)}$ . Notice that the integer  $l(g)$  and Cartan's projection  $a^+(g)$  are uniquely determined by  $g$ .

Let us write  $a^+(g)$  as a matrix of  $\mathrm{O}(2, n)$ :

$$a^+(g) = \begin{pmatrix} e^{\lambda(g)} & & & & & \\ & e^{\mu(g)} & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & e^{-\mu(g)} \\ & & & & & & e^{-\lambda(g)} \end{pmatrix}.$$

The reals  $\lambda(g) \geq \mu(g) \geq 0$  are called the *distorsions* of the element  $g$ .

### 3.2 Sequences tending to infinity

Let us consider now a sequence  $(g_k)$  in  $\widetilde{\mathrm{O}(2, n)}$ , which tends to infinity (i.e. leaves every compact subset of  $\widetilde{\mathrm{O}(2, n)}$ ). Looking, if necessary, at a subsequence we can suppose:

a) The four sequences  $l(g_k)$ ,  $\lambda(g_k)$ ,  $\mu(g_k)$  and  $\delta(g_k) = \lambda(g_k) - \mu(g_k)$  converge respectively to  $l_\infty$ ,  $\lambda_\infty$ ,  $\mu_\infty$  and  $\delta_\infty$  in  $\mathbb{R}$ .

b) Compact factors in the Cartan decomposition of  $(g_k)$  both admit a limit in  $\bar{K}$  as  $k$  tends to infinity.

Under these assumptions, the sequence  $(g_k)$  falls into one of the four following categories:

(i)  $|l_\infty| = +\infty$ . The sequence  $(g_k)$  is then said to be *proper*. For the three other cases  $l_\infty$  is finite.

(ii) If  $\mu_\infty \neq +\infty$ , the sequence  $(g_k)$  is said to have *bounded distortions*.

(iii) If  $\lambda_\infty = \mu_\infty = +\infty$  and  $\delta_\infty$  is finite, the sequence  $(g_k)$  is said to have *balanced distortions*.

(iv) If  $\lambda_\infty = \mu_\infty = \delta_\infty = +\infty$ , the sequence  $(g_k)$  is said to have *mixed distortions*.

### 3.3 The limit set of a complete anti-de Sitter structure

In [Fr2], we studied extensively the dynamical behaviour on  $\text{Ein}_n$  of sequences tending to infinity in  $\text{O}(2, n)$ . This description can be adapted to sequences of  $\widetilde{\text{O}(2, n)}$  acting on  $\widetilde{\text{AdS}}_{n+1}$  (resp.  $\widetilde{\text{Ein}}_n$ ), and we refer to [Fr2] for the proofs. Here we state only the few properties which will be relevant for our purpose. First, we recall the following definition.

**Definition 1** (Properness). Let  $(g_k)$  be a sequence of  $\widetilde{\text{O}(2, n)}$  tending to infinity and  $\Omega$  be an open subset of  $\widetilde{\text{AdS}}_{n+1}$ , which is left invariant under the action of  $(g_k)$ . The sequence  $(g_k)$  acts properly on  $\Omega$  if for any pair  $(K, K')$  of compact subsets of  $\Omega$ ,  $g_k(K) \cap K' = \emptyset$  for all but a finite number of  $k$ 's.

**Proposition 1.** A sequence  $(g_k)$  of  $\widetilde{\text{O}(2, n)}$  tending to infinity acts properly on  $\widetilde{\text{AdS}}_{n+1}$  if and only if it is proper.

In Riemannian signature, Ascoli's theorem ensures that a sequence of isometries of a manifold  $X$  always acts properly on  $X$ . This is no longer true in Lorentzian geometry, and this is what makes Lorentzian dynamics richer. For example, a sequence of  $\widetilde{\text{O}(2, n)}$  can act non properly on  $\widetilde{\text{AdS}}_{n+1}$ . The next proposition is a useful characterization of sequences in  $\widetilde{\text{O}(2, n)}$  acting properly on  $\widetilde{\text{AdS}}_{n+1}$ .

**Proposition 2.** A sequence  $(g_k)$  of  $\widetilde{\text{O}(2, n)}$  acts properly on  $\widetilde{\text{AdS}}_{n+1}$  if and only if  $(g_k)$  does not have a subsequence with bounded distortion. In particular, a discrete subgroup  $\Gamma \subset \widetilde{\text{O}(2, n)}$  acts properly discontinuously on  $\widetilde{\text{AdS}}_{n+1}$  if and only if  $\Gamma$  does not contain an infinite sequence with bounded distortion.

The next step is the dynamical description of sequences with mixed or balanced distortions.

**Proposition 3.** Let  $(g_k)$  be an infinite sequence of  $\widetilde{\text{O}(2, n)}$  with bounded or mixed distortions. There are two lightlike geodesics  $\Delta^+(g_k)$  and  $\Delta^-(g_k)$  in  $\widetilde{\text{Ein}}_n$ , called attracting and repelling geodesics, which are uniquely determined by the sequence  $(g_k)$  such

that: For any compact subset  $K \subset \overline{\text{AdS}}_{n+1} \setminus \{\Delta^-(g_k)\}$  (resp.  $K \subset \overline{\text{AdS}}_{n+1} \setminus \{\Delta^+(g_k)\}$ ),  $g_k(K)$  (resp.  $g_k^{-1}(K)$ ) tends (for the Hausdorff topology) to a closed subset of  $\Delta^+(g_k)$  (resp.  $\Delta^-(g_k)$ ).

Let  $M$  be a manifold endowed with a *complete* anti-de Sitter metric  $g$ . The structure  $(M, g)$  is obtained as the quotient of the space  $\widetilde{\text{AdS}}_{n+1}$  by a discrete subgroup  $\Gamma \subset \text{O}(2, n)$  acting properly discontinuously on  $\widetilde{\text{AdS}}_{n+1}$ . By Proposition 2, the group  $\Gamma$  does not contain any sequence  $(\gamma_k)$  with bounded distortion. Let  $\mathcal{S}$  be the set of all sequences  $(\gamma_k) \subset \Gamma$ , which have either mixed or balanced distortion. Notice that any infinite sequence of  $\Gamma$  has a subsequence in  $\mathcal{S}$ . We define the *limit set* of the group  $\Gamma$  in the following way:

$$\Lambda_\Gamma = \overline{\bigcup_{(\gamma_k) \in \mathcal{S}} \Delta^+(\gamma_k) \cup \Delta^-(\gamma_k)}.$$

It is clear that the set  $\Lambda_\Gamma$  is closed,  $\Gamma$ -invariant and included in  $\widetilde{\text{Ein}}_n$ . Let us call  $\Omega_\Gamma$  the complementary of  $\Lambda_\Gamma$  in  $\overline{\text{AdS}}_{n+1}$ . We deduce easily from Proposition 3:

**Proposition 4.** *The action of  $\Gamma$  on  $\Omega_\Gamma$  is proper and discontinuous.*

For a more general definition of the limit set of a discrete subgroup  $\Gamma \subset \text{O}(2, n)$  we refer to [Fr2]. Other definitions of the limit set, in the framework of linear discrete groups, can be found in [B].

## 4 The conformal boundary completion of a complete anti-de Sitter structure

### 4.1 Building the boundary

Let  $(M, g)$  be a complete anti-de Sitter structure obtained as the quotient  $\widetilde{\text{AdS}}_{n+1} / \Gamma$ . By proposition 4, the quotient  $\bar{M} = \Omega_\Gamma / \Gamma$  is a manifold with boundary. The interior of  $\bar{M}$  is the complete anti-de Sitter structure  $\widetilde{\text{AdS}}_{n+1} / \Gamma$  we started with. The boundary  $\partial M$  is the conformally flat Lorentzian manifold  $(\Omega_\Gamma \cap \widetilde{\text{Ein}}_n) / \Gamma$ . In fact,  $\bar{M}$  inherits from  $\Omega_\Gamma$  a canonical conformal class of Lorentzian metrics. Let us pick some metric  $\bar{g}$  in this class. We lift  $\bar{g}$  (resp.  $g$ ) into a  $\Gamma$ -invariant  $\bar{h}$  (resp.  $h$ ) on  $\Omega_\Gamma$  (resp.  $\widetilde{\text{AdS}}_{n+1}$ ). The metric  $h$  is nothing else than the canonical metric  $g_{\text{can}}$  of  $\widetilde{\text{AdS}}_{n+1}$ . For this canonical metric there is a well-known conformal boundary. So, there is a smooth metric  $\bar{h}'$  on  $\overline{\text{AdS}}_{n+1}$  and a function  $\beta$  on  $\widetilde{\text{AdS}}_{n+1}$  such that  $\bar{h}' = \beta^2 h$ . Moreover,  $\widetilde{\text{Ein}}_n = \beta^{-1}(\{0\})$  and  $d\beta \neq 0$  on  $\widetilde{\text{Ein}}_n$ . On the other hand,  $\bar{h}$  and  $\bar{h}'$  are in the same conformal class so that  $\bar{h} = \alpha^2 \bar{h}'$  on  $\Omega_\Gamma$ . We thus get  $\bar{h} = \rho^2 h$  on  $\widetilde{\text{AdS}}_{n+1}$ , with  $\rho = \alpha\beta$ . The function  $\rho$  is smooth on  $\Omega_\Gamma$ ,  $\rho^{-1}(\{0\}) = \Omega_\Gamma \cap \widetilde{\text{Ein}}_n$  and  $d\rho \neq 0$  on  $\Omega_\Gamma \cap \widetilde{\text{Ein}}_n$ . Moreover,

it has to be  $\Gamma$ -invariant since this is the case for  $\bar{h}$  and  $h$ . Thus,  $\rho$  induces a smooth function  $\bar{\rho}$  on  $\bar{M}$  such that  $\bar{g} = \bar{\rho}^2 g$  on  $M$ . Clearly  $\bar{\rho}^{-1}(\{0\}) = \partial M$  and  $d\bar{\rho} \neq 0$  on  $\partial M$ . This proves that  $\bar{M}$  is a *conformal boundary completion* of  $(M, g)$ .

**Remark.** Of course, it can happen that for some complete structure  $\widetilde{\text{AdS}}_{n+1}/\Gamma$ ,  $\Lambda_\Gamma = \widetilde{\text{Ein}}_n$ . In this case the previous construction yields an empty conformal boundary  $\partial M$ .

## 4.2 $(G, X, \partial X)$ -structures

Let  $\bar{X}$  be a manifold with boundary  $\partial X$ , and interior  $X$ . We write  $\bar{X} = X \cup \partial X$ . Suppose that some Lie group  $G$  acts transitively on  $X \cup \partial X$ . Suppose moreover that the action of  $G$  is strongly effective. This means that two elements  $g_1$  and  $g_2$  of  $G$  acting in the same way on an open subset  $U \subset X$  are in fact equal. Then we can define the notion of a  $(G, X, \partial X)$ -structure on a manifold  $\bar{M}$  with boundary  $\partial M$  and interior  $M$ : It is given by

- an open covering  $(U_i)_{i \in I}$  of  $\bar{M}$ ,
- a family of embeddings  $\phi_i : U_i \rightarrow \bar{X}$  respecting the boundaries (i.e.  $\phi_i(U_i \cap \partial M) = \phi_i(U_i) \cap \partial X$ ),
- a family  $(g_{ji})_{i,j \in I}$  of elements of  $G$  such that for all  $i, j \in I$ , the map  $\phi_j \circ \phi_i^{-1}$  agrees with  $g_{ji}$  on  $\phi_i(U_i \cap U_j)$ .

With this definition the conformal boundary completion of a complete anti-de Sitter structure, as defined above, is endowed with a  $(\widetilde{\text{O}(2, n)}, \widetilde{\text{AdS}}_{n+1}, \widetilde{\text{Ein}}_n)$ -structure.

## 4.3 Main properties of the conformal boundary completion for a complete anti-de Sitter structure

The first and perhaps most important property is that, under some natural assumptions, there is essentially a unique conformal boundary completion for a given complete anti-de Sitter structure. Since we are dealing, a priori, with non compact conformal boundary completions, it is natural to introduce the notion of maximal completion.

**Definition 2.** Let  $(\bar{M}, [\bar{g}])$  and  $(\bar{N}, [\bar{h}])$  two conformal boundary completions of a space-time  $(M, g)$ . We write  $(\bar{M}, [\bar{g}]) \subset (\bar{N}, [\bar{h}])$  if there is a conformal embedding of  $(\bar{M}, [\bar{g}])$  into  $(\bar{N}, [\bar{h}])$  which maps  $\partial M$  into  $\partial N$ . A maximal element for the relation  $\subset$  is called a *maximal conformal boundary completion* of  $(M, g)$ .

We can now prove:

**Theorem 1.** *Let  $(M, g)$  a complete anti-de Sitter structure. If  $(\bar{M}, [\bar{g}])$  and  $(\bar{N}, [\bar{h}])$  are two maximal conformal boundary completions of  $M$ , and if moreover  $\bar{M}$  and  $\bar{N}$  both admit a  $(\widetilde{\mathrm{O}(2, n)}, \widetilde{\mathrm{AdS}}_{n+1}, \widetilde{\mathrm{Ein}}_n)$ -structure, then they are isomorphic (i.e. there is a diffeomorphism of manifolds with boundary between  $\bar{M}$  and  $\bar{N}$  which is conformal).*

*Proof.* We write  $\bar{M} = M \cup \partial M$  and denote by  $\widetilde{M \cup \partial M}$  the universal cover of  $\bar{M}$ . As for classical  $(G, X)$ -structures,  $(G, X, \partial X)$ -structures admit a developing map  $\delta$  and a holonomy morphism  $\rho$  (see [Th]). In our case  $\delta$  is a local diffeomorphism from  $\widetilde{M \cup \partial M}$  to  $\widetilde{\mathrm{AdS}}_{n+1}$ , which sends the boundary of  $\widetilde{M \cup \partial M}$  into  $\widetilde{\mathrm{Ein}}_n$ . The interior of  $\widetilde{M \cup \partial M}$  is just  $\tilde{M}$ , the universal cover of  $M$ . The restriction of  $\delta$  to  $\tilde{M}$  is the developing map of the anti-de Sitter structure on  $M$ . Since  $(M, g)$  is complete,  $\delta|_{\tilde{M}}$  is injective. Since  $\delta$  is a local diffeomorphism, we infer that  $\delta$  is injective on the whole  $\widetilde{M \cup \partial M}$ . Thus,  $(\bar{M}, [\bar{g}])$  is obtained as a quotient  $\Omega / \Gamma$ , where  $\Omega$  is an open subset of  $\widetilde{\mathrm{AdS}}_{n+1}$  and  $\Gamma$  is the discrete group of  $\mathrm{O}(2, n)$  such that  $(M, g) = \widetilde{\mathrm{AdS}}_{n+1} / \Gamma$ . We now prove the following.

**Lemma 1.** *If a discrete subgroup  $\Gamma \subset \widetilde{\mathrm{O}(2, n)}$  acts properly on some open subset  $\Omega$ , with  $\widetilde{\mathrm{AdS}}_{n+1} \subset \Omega \subset \widetilde{\mathrm{AdS}}_{n+1}$ , then  $\Omega \subset \Omega_\Gamma$ .*

*Proof.* Since  $\widetilde{\mathrm{AdS}}_{n+1} \subset \Omega$  and  $\Gamma$  acts properly on  $\Omega$ , we know from Section 3.3 that  $\Gamma$  has a limit set  $\Lambda_\Gamma$  and so  $\Omega_\Gamma$  is well defined. Now let  $(\gamma_k)$  be a sequence of  $\Gamma$  with balanced or mixed distortions. We call  $\Delta^+(\gamma_k)$  and  $\Delta^-(\gamma_k)$  its attracting and repelling lightlike geodesics. We project  $(\gamma_k)$  on  $\tilde{\gamma}_k \in \mathrm{O}(2, n)$  and  $\Delta^+(\gamma_k)$  (resp.  $\Delta^-(\gamma_k)$ ) on  $\tilde{\Delta}^+ \subset \mathrm{Ein}_n$  (resp.  $\tilde{\Delta}^- \subset \mathrm{Ein}_n$ ). Let  $x \in \tilde{\Delta}^-$  and  $U$  be a small neighbourhood of  $x$  in  $\mathrm{Ein}_{n+1}$ . In [Fr2] (Sections 3.2.1, 3.2.3) we proved that the Hausdorff limit  $\lim_{k \rightarrow +\infty} \tilde{\gamma}_k(U)$  is either a lightcone (of  $\mathrm{Ein}_{n+1}$ ) or some closed subset with non-empty interior. In any case this limit meets the complementary of  $\mathrm{Ein}_n$  in  $\mathrm{Ein}_{n+1}$ . Lifting all these results to  $\widetilde{\mathrm{Ein}}_n$  we get that if  $x \in \tilde{\Delta}^-(\gamma_k)$  and  $U$  is a small neighbourhood of  $x$  in  $\widetilde{\mathrm{AdS}}_{n+1}$ , then the Hausdorff limit of  $\tilde{\gamma}_k(U)$  meets  $\widetilde{\mathrm{AdS}}_{n+1}$ . So if  $\Omega$  meets  $\tilde{\Delta}^-(\gamma_k)$ , the action can not be proper on  $\Omega$  since  $\widetilde{\mathrm{AdS}}_{n+1} \subset \Omega$  by hypothesis. Looking at  $(\gamma_k^{-1})$  one proves also that  $\Omega$  can not meet  $\tilde{\Delta}^+(\gamma_k)$ . We infer that  $\Lambda_\Gamma \cap \Omega = \emptyset$  and  $\Omega \subset \Omega_\Gamma$ .  $\square$

The previous result and the maximality hypothesis on  $(\bar{M}, [\bar{g}])$  yields that  $(\bar{M}, [\bar{g}]) = \Omega_\Gamma / \Gamma$ . So the structure of  $(\bar{M}, [\bar{g}])$  is entirely defined under the two hypothesis that  $(\bar{M}, [\bar{g}])$  is maximal and shares a  $(\widetilde{\mathrm{O}(2, n)}, \widetilde{\mathrm{AdS}}_{n+1}, \widetilde{\mathrm{Ein}}_n)$ -structure.  $\square$

From now, when we will speak of the conformal boundary completion of a complete anti-de Sitter structure  $(M, g) = \widetilde{\mathrm{AdS}}_{n+1} / \Gamma$ , we will always mean  $(\bar{M}, [\bar{g}]) = \Omega_\Gamma / \Gamma$  (this is justified by Theorem 1).

**Theorem 2.** *Let  $(M, g) = \widetilde{\text{AdS}}_{n+1}/\Gamma_1$  and  $(N, h) = \widetilde{\text{AdS}}_{n+1}/\Gamma_2$  be two complete anti-de Sitter structures of dimension  $\geq 4$  with conformal boundary completions  $(\bar{M}, [\bar{g}])$  and  $(\bar{N}, [\bar{h}])$ . Suppose that the manifolds  $(\partial M, [\bar{g}]|_{\partial M})$  and  $(\partial N, [\bar{h}]|_{\partial N})$  are conformally equivalent, then  $(M, g)$  and  $(N, h)$  are isometric.*

*Proof.* The structures  $(\partial M, [\bar{g}]|_{\partial M})$  and  $(\partial N, [\bar{h}]|_{\partial N})$  are Kleinian, obtained as quotients  $(\Omega_{\Gamma_1} \cap \widetilde{\text{Ein}}_n)/\Gamma_1$  and  $(\Omega_{\Gamma_2} \cap \widetilde{\text{Ein}}_n)/\Gamma_2$  respectively. A conformal diffeomorphism  $\phi$  from  $(\partial M, [\bar{g}]|_{\partial M})$  to  $(\partial N, [\bar{h}]|_{\partial N})$  yields a conformal transformation  $\phi$  between  $\Omega_{\Gamma_1} \cap \widetilde{\text{Ein}}_n$  and  $\Omega_{\Gamma_2} \cap \widetilde{\text{Ein}}_n$  such that  $\phi\Gamma_1 = \Gamma_2\phi$ . But in dimension  $\geq 3$  any conformal transformation between  $\Omega_1$  and  $\Omega_2$  is the restriction of some element of  $\widetilde{\text{O}(2, n)}$  by Liouville's theorem (see [S], [Fr3]). So  $\phi \in \widetilde{\text{O}(2, n)}$  and  $\phi\Gamma_1\phi^{-1} = \Gamma_2$ . Thus, since  $(M, g)$  and  $(N, h)$  are respectively  $\widetilde{\text{AdS}}_{n+1}/\Gamma_1$  and  $\widetilde{\text{AdS}}_{n+1}/\Gamma_2$ ,  $(M, g)$  and  $(N, h)$  are isometric.  $\square$

**Proposition 5.** *Let  $(M, g)$  be a complete anti-de Sitter structure and  $(\bar{M}, [\bar{g}])$  be its conformal boundary completion. Then any isometry of  $(M, g)$  extends to a diffeomorphism of  $\bar{M}$  acting conformally on  $(\partial M, [\bar{g}]|_{\partial M})$ . Reciprocally, any conformal transformation of  $(\partial M, [\bar{g}]|_{\partial N})$  extends to a diffeomorphism of  $\bar{M}$  acting isometrically on  $(M, g)$ .*

*Proof.* If we write  $(M, g)$  as a quotient  $\widetilde{\text{AdS}}_{n+1}/\Gamma$ , the structure  $(\partial M, [\bar{g}]|_{\partial M})$  is a Kleinian structure, obtained as the quotient  $(\Omega_{\Gamma} \cap \widetilde{\text{Ein}}_n)/\Gamma$ . Any isometry of  $(M, g)$  is induced by an element  $\phi \in \widetilde{\text{O}(2, n)}$  satisfying  $\phi\Gamma\phi^{-1} = \Gamma$ . But since  $\phi(\Lambda_{\Gamma}) = \Lambda_{\phi\Gamma\phi^{-1}}$ , we get that  $\phi(\Omega_{\Gamma}) = \Omega_{\Gamma}$ , and  $\phi$  induces a conformal diffeomorphism of  $(\bar{M}, [\bar{g}])$ . In particular, it induces a conformal diffeomorphism of  $(\partial M, [\bar{g}]|_{\partial M})$ . Reciprocally, a conformal transformation of  $(\partial M, [\bar{g}]|_{\partial M})$  is induced by some  $\phi \in \widetilde{\text{O}(2, n)}$  such that  $\phi\Gamma\phi^{-1} = \Gamma$  and  $\phi(\Omega_{\Gamma} \cap \widetilde{\text{Ein}}_n) = \Omega_{\Gamma} \cap \widetilde{\text{Ein}}_n$ . But then  $\phi(\Omega_{\Gamma}) = \Omega_{\Gamma}$ , and  $\phi$  induces a diffeomorphism of  $\bar{M}$ . This diffeomorphism is moreover an isometry of  $(M, g)$  since  $\phi(\widetilde{\text{AdS}}_{n+1}) = \widetilde{\text{AdS}}_{n+1}$  and  $\phi\Gamma\phi^{-1} = \Gamma$ .  $\square$

## 5 Illustration

### 5.1 The example of Lorentzian Schottky groups

Let  $\Delta_i^{\pm}$ ,  $i = 1, \dots, g$ , be a collection of  $2g$  pairwise disjoint lightlike geodesics in  $\widetilde{\text{Ein}}_n$ . Then there exist  $g$  elements  $\gamma_1, \dots, \gamma_g$  in  $\widetilde{\text{O}(2, n)}$  and a family  $U_i^{\pm}$  of  $2g$  open subsets of  $\widetilde{\text{AdS}}_{n+1}$  with the following properties:

- (i) For all  $1 \leq i \leq g$ ,  $U_i^{\pm}$  is a tubular neighbourhood of  $\Delta_i^{\pm}$ .
- (ii) The closures of the  $U_i^{\pm}$ 's are pairwise disjoint.

- (iii) For all  $1 \leq i \leq g$ ,  $\gamma_i$  (resp.  $\gamma_i^{-1}$ ) maps the exterior of  $U_i^-$  (resp.  $U_i^+$ ) into  $\widetilde{\text{AdS}}_{n+1}$  on the closure of  $U_i^+$  (resp.  $U_i^-$ ).

The group  $\Gamma$  generated by the elements  $\gamma_1, \dots, \gamma_g$  is then a free group called Lorentzian Schottky group. We studied such groups in [Fr2]. In particular, we proved:

**Theorem 3.** *Let  $\Gamma \subset \widetilde{\text{O}(2, n)}$  ( $n \geq 3$ ) be a Lorentzian Schottky group with  $g$  generators ( $g \geq 2$ ). Then:*

- (i) *The action of  $\Gamma$  on  $\widetilde{\text{AdS}}_{n+1}$  is proper.*
- (ii) *The limit set  $\Lambda_\Gamma$  is a lamination by lightlike geodesics. Topologically it is a product of  $\mathbb{R}$  by a Cantor set.*
- (iii) *The conformal boundary of the complete anti-de Sitter space-time  $\widetilde{\text{AdS}}_{n+1}/\Gamma$  is diffeomorphic to  $\mathbb{R} \times (\mathbb{S}^1 \times \mathbb{S}^{n-2})^{\sharp(g-1)}$ . Here  $(\mathbb{S}^1 \times \mathbb{S}^{n-2})^{\sharp(g-1)}$  denotes the connected sum of  $g-1$  copies of  $(\mathbb{S}^1 \times \mathbb{S}^{n-2})$ .*

For  $n = 2$  there is an alternative statement to this theorem, but this time the conformal boundary is not connected: it is a finite union of cylinders  $\mathbb{R} \times \mathbb{S}^1$ .

## 5.2 Other examples in odd dimension

The group  $\text{U}(1, n)$ , preserving the Lorentzian hermitian form  $-|z_1|^2 + |z_2|^2 + \dots + |z_n|^2$ , admits a canonical injection in  $\text{O}(2, 2n)$ . This yields an injection of the universal covers  $\widetilde{\text{U}(1, n)} \subset \widetilde{\text{O}(2, 2n)}$ . In fact, there is a natural fibration of  $\widetilde{\text{Ein}}_{2n}$  over the canonical flat CR-sphere  $\mathbb{S}^{2n-1}$ , the fibers of which are lightlike geodesics. The group  $\widetilde{\text{U}(1, n)}$  is exactly the subgroup of  $\widetilde{\text{O}(2, 2n)}$  preserving this fibration. On the other hand, as first observed by R. Kulkarni,  $\text{U}(1, n)$  acts properly on  $\widetilde{\text{AdS}}_{2n+1}$ . Let  $(M, g)$  be a complete anti-de Sitter structure obtained as a quotient  $\text{AdS}_{2n+1}/\Gamma$  with  $\Gamma \subset \widetilde{\text{U}(1, n)}$ . The group  $\Gamma$  projects as a subgroup  $\bar{\Gamma} \subset \text{U}(1, n)$ . The group  $\bar{\Gamma}$  acts on  $\mathbb{S}^{2n-1}$  as a convergence group. In particular, it has a domain of discontinuity  $\bar{\Omega}_{\bar{\Gamma}}$  (which turns out to be the projection of  $\Omega_\Gamma \cap \text{Ein}_{2n}$ ). We obtain that the conformal boundary of  $(M, g)$  is a fibration by lightlike geodesics over the CR-flat Kleinian manifold  $\bar{\Omega}_{\bar{\Gamma}}/\bar{G}$  (we refer to [Go] for an account on CR-flat Kleinian manifolds). In other words, the structure  $(M, g)$  fibers over a complete complex hyperbolic manifold, the boundary of which is the CR-manifold  $\bar{\Omega}_{\bar{\Gamma}}/\bar{\Gamma}$ . We just saw that the fibration extends to the respective boundaries.

**Acknowledgements.** I would like to thank A. Zeghib for useful comments on this text.

## References

- [A] M. Anderson, Einstein metrics with prescribed conformal infinity on 4-manifolds, preprint, arXiv:math.DG/0105243.
- [B] Y. Benoist, Propriétés asymptotiques des groupes linéaires, *Geom. Funct. Anal.* 7 (1997), 1–47.
- [Bi] O. Biquard Métriques d'Einstein asymptotiquement symétriques, *Astérisque* 265 (2000).
- [BTZ1] M. Bañados, C. Teitelboim, J. Zanelli, Black hole in three-dimensional space-time, *Phys. Rev. Lett.* 69 (1992), 1849–1851.
- [BTZ2] M. Bañados, M. Henneaux, C. Teitelboim, J. Zanelli, Geometry of the 2 + 1 black hole, *Phys. Rev. D* (3) 48 (1993), no. 4, 1506–1525.
- [CK] M. Cahen, Y. Kerbrat, Domaines symétriques des quadriques projectives, *J. Math. Pures Appl.* (9) 62 (1983), no. 3, 327–348.
- [C-H] P. Chrusciel, M. Herzlich, The mass of asymptotically hyperbolic Riemannian manifolds, *Pacific J. Math.* 212 (2003), 231–264.
- [Fr1] C. Frances, Géométrie et dynamique lorentziennes conformes, Thèse, ENS-Lyon, 2002, available at <http://www.umpa.ens-lyon.fr/~cfrances/>.
- [Fr2] C. Frances, Lorentzian Kleinian groups, preprint, 2003, available at <http://www.umpa.ens-lyon.fr/~cfrances/>.
- [Fr3] C. Frances, Les structures conformes pseudo-riemanniennes vues comme structures rigides à l'ordre 2, preprint, 2002, available at <http://www.umpa.ens-lyon.fr/~cfrances/>.
- [Go] W. Goldman, Complex hyperbolic Kleinian groups, in: *Complex geometry*, Proceedings of the Osaka International Conference, G. Komatsu and Y. Sakane (Eds.), Marcel Dekker, Inc., New York 1992, 31–52.
- [HE] S. Hawking, G. Ellis, *The large scale structure of universe*, Cambridge University Press, Cambridge 1973.
- [IW] A. Iozzi, D. Witte, Cartan-decomposition subgroups of  $SU(2, n)$ , *J. Lie Theory* 11 (2001), 505–543.
- [O'N] B. O'Neill, *Semi-Riemannian Geometry. With applications to relativity*, Pure Appl. Math. 103, Academic Press, Inc., New York 1983.
- [P] R. Penrose, Conformal treatment of infinity, in: *Relativité, Groupes et Topologie* (Lectures, Les Houches, 1963 Summer School of Theoret. Phys., Univ. Grenoble), Gordon and Breach, New York 1964, 563–584.
- [S] I. Segal, *Mathematical cosmology and extragalactic astronomy*, Pure Appl. Math. 68, Academic Press, Inc., New York–London 1976.
- [Th] W. Thurston, *Three-dimensional geometry and topology*, vol. 1, ed. by Silvio Levy, Princeton University Press, Princeton 1997.



- [Wo] J. Wolf, *Spaces of constant curvature*, fifth edition, Publish or Perish Inc., Houston, TX, 1984.
- [Ze] A. Zeghib On closed anti de Sitter space-times, *Math. Ann.* 310 (1998), 695–716.

# Supersymmetric AdS backgrounds in string and M-theory

*Jerome P. Gauntlett, Dario Martelli,  
James Sparks and Daniel Waldram*

*Blackett Laboratory, Imperial College  
London, SW7 2BZ, U.K.  
email: j.gauntlett, d.waldram@imperial.ac.uk*

*Department of Physics, CERN Theory Division  
1211 Geneva 23, Switzerland  
email: dario.martelli@cern.ch*

*Department of Mathematics, Harvard University  
One Oxford Street, Cambridge, MA 02318, U.S.A.  
and  
Jefferson Physical Laboratory, Harvard University  
Cambridge, MA 02318, U.S.A.  
email: sparks@math.harvard.edu*

**Abstract.** We first present a short review of general supersymmetric compactifications in string and M-theory using the language of  $G$ -structures and intrinsic torsion. We then summarize recent work on the generic conditions for supersymmetric  $\text{AdS}_5$  backgrounds in M-theory and the construction of classes of new solutions. Turning to  $\text{AdS}_5$  compactifications in type IIB, we summarize the construction of an infinite class of new Sasaki–Einstein manifolds in dimension  $2k + 3$  given a positive curvature Kähler–Einstein base manifold in dimension  $2k$ . For  $k = 1$  these describe new supergravity duals for  $\mathcal{N} = 1$  superconformal field theories with both rational and irrational R-charges and central charge. We also present a generalization of this construction, that has not appeared elsewhere in the literature, to the case where the base is a product of Kähler–Einstein manifolds.

## 1 Introduction

In this paper we aim to review first the general framework of supersymmetric solutions of string or M-theory, where spacetime is a product  $E \times X$  of an external manifold  $E$  and an internal manifold  $X$ , and then, secondly, two interesting classes of examples where

$E$  is five-dimensional anti-de Sitter ( $\text{AdS}_5$ ) space and  $X$  is five- or six-dimensional. This latter work was first presented in three papers, [42], [43] and [44]. Such backgrounds are central in string theory first, when  $E$  is four-dimensional Minkowski space, as a way to construct semi-realistic supersymmetric models of particle physics, and second, when  $E$  is an AdS space, as gravitation duals of quantum conformal field theories, via the AdS-CFT correspondence (for a review see [1]).

We consider string or M-theory in the low-energy supergravity limit where the condition for a supersymmetric solution requires the existence of a constant spinor with respect to a particular Clifford algebra-valued connection  $D^X$ , perhaps supplemented with additional algebraic conditions on the spinor. When certain fields, so-called  $p$ -form fluxes, in the supergravity are zero,  $D^X$  is equal to the Levi-Civita connection and hence supersymmetry translates into a condition of special holonomy. However, in many cases one wants to include non-trivial flux. In the first part of the paper we review how this translates into the existence of a  $G$ -structure  $P$  and how the fluxes are encoded in the intrinsic torsion of  $P$ . We also comment on the relation to generalized holonomy and generalized calibrations. By way of an example we concentrate on the case of  $d = 11$  supergravity on a seven-dimensional  $X$  with  $\text{SU}(3)$ -structure, and type IIB supergravity on a six-dimensional  $X$  with  $\text{SU}(3)$ -structure and only five-form flux excited.

The second part of the paper, based on [42], discusses first the general conditions on the geometry of  $X$  in supersymmetric  $\text{AdS}_5 \times X$  solutions of  $d = 11$  supergravity, and second a large family of explicit regular solutions of this form characterized by  $X$  being complex. Previously, a surprisingly small number of explicit solutions were known. Most notable was that of Maldacena and Nuñez [74] describing the near horizon limit of fivebranes wrapping constant curvature holomorphic curves in Calabi–Yau three-folds. The new solutions can be viewed as corresponding to a more general type of embedded holomorphic curve. They fall into two classes where  $X$  is a fibration of a two-sphere over either a four-dimensional Kähler–Einstein manifold or a product of constant-curvature Riemann surfaces.

The third part of the paper relates to Sasaki–Einstein (SE) manifolds. These arise as the internal manifold  $X$  in supersymmetric type IIB  $\text{AdS}_5 \times X$  solutions. We review a new construction ([43], [44]) of an infinite class of SE manifolds in any dimension  $n = 2k + 3$  based on an underlying  $2k$ -dimensional positive curvature Kähler–Einstein manifold. All SE spaces have a constant norm Killing vector  $K$  (see, for instance, [15] and [29]) and can be characterized by whether the orbits of  $K$  are closed (so-called regular and quasi-regular cases) or not (the irregular case). The new class of solutions includes quasi-regular and irregular cases. Again, previously, surprisingly few explicit SE metrics were known: the homogeneous regular cases have been classified [14]; several quasi-regular examples had been constructed using algebraic geometry techniques but without an explicit metric; and no irregular examples were known. Finally we give a straightforward extension of the construction to the case where the underlying manifold is a product of Kähler–Einstein spaces. This is new material and leads to new  $\text{AdS}_4 \times X_7$  solutions of M-theory.

The history of considering supersymmetric backgrounds of supergravity theories with non-trivial fluxes is a comparatively long one. The use of  $G$ -structures to classify such backgrounds was first proposed in [41]. This was based partly on earlier work by Friedrich and Ivanov [35], though these authors did not consider the supergravity equations of motion. The relationship between background supersymmetry conditions and generalized calibrations [56] was first discussed slightly earlier in [39] and shown to be generic in [37], [38].

These techniques have subsequently been developed and extended in a number of directions. First, one can use  $G$ -structures to classify *all* supersymmetric solutions of a given supergravity theory. This has been carried out for the most generic case of a single preserved supersymmetry in  $d = 11$  supergravity in [37]. A similar classification has now also been worked out for simpler supergravity theories in four [17], five [45], six [57] and seven [20] dimensions. Note that this work extends older work of Tod [90] which classified supersymmetric solutions of four-dimensional supergravity using techniques specific to four dimensions. An important open problem in, for instance, the  $d = 11$  case is to refine the classification presented in [37] and determine the extra conditions required for solutions to preserve more than one supersymmetry. There has been some recent progress on this using  $G$ -structures, partly implementing some suggestions in [37], in the context of seven [73] and eleven dimensions [47]. Note that the case of maximal supersymmetry can be analysed using different techniques and this has been carried out for type IIB and  $d = 11$  supergravity in [31]. A quite different attempt at classification, first advocated in [28] and subsequently studied in [62], is to use the notion of “generalized holonomy”. We comment on the relation to  $G$ -structures in the next section.

A second application is to use  $G$ -structures to analyse supersymmetric “flux compactifications” in string theory. These are supersymmetric backgrounds where the external space  $E$  is flat Minkowski space and  $X$  is often, but not always, compact. This is a large field with a rich literature, starting with that of Strominger [86] and Hull [61] in the context of the heterotic string (see also [27]). More recently, starting with Polchinski and Strominger [82] as well as [3], several authors have analyzed flux backgrounds for the special case where  $X$  is a special holonomy manifold (for early work see [4], [77], [52], [26] and [87]) and the resulting low-energy effective theories on  $E$  (also a large field, see, for instance, the references in [55] or [19]). Let us concentrate on the use of  $G$ -structure techniques to analyse cases when  $X$  does not have special holonomy. In [55] it was argued that the mirror of a Calabi–Yau threefold with three-form  $H$ -flux is a manifold with a “half-flat”  $SU(3)$ -structure. Further work in this direction appears in [30]. For the heterotic string, Strominger’s and Hull’s results imply  $X$  has a non-Kähler  $SU(3)$  structure and these have been analyzed in [38], [19], [48], [5], [6], [18], [65]. Such flux compactifications have only  $H$ -flux, and these were completely classified, including type II backgrounds, in [38]. (Note that ref. [38] corrects a sign in [86], disqualifying the putative Iwasawa solutions in [19].) Flux compactifications on more general  $SU(3)$ -structures have been considered in [53], [75], [34], [78]. General type II compactifications with more general fluxes

have been addressed, for instance, in [50], [33], [32], [7], [24], [84], [8]. General  $d = 11$  flux compactifications have been discussed in terms of  $G$ -structures in several papers ([25], [67], [10], [66], [76], [9], [72], [11], [60]).

A third connected application is to spacetime solutions dual to supersymmetric field theories via the AdS-CFT correspondence. The basic case of interest is when  $E$  is AdS since the solution is then dual to a supersymmetric conformal field theory. There are more general kinds of solutions, however, that are dual to other types of field theories, as well as to renormalisation group flows (see the review [1]). Again this is very large field. Aside from the initial paper [41] (which focussed on solutions dual to “little string theories”) and the work [42], [43], [44] on which this paper is based,  $G$ -structures have been used to analyze AdS solutions in refs. [76] and [72]. Very recently an interesting class of half-supersymmetric solutions has been found [71]. Note that there is also a related approach to finding special sub-classes of solutions initiated by Warner and collaborators (see for instance [49]).

Finally, we comment on some work related to  $G$ -structure classifications and generalised calibrations. Calibrations are important in string theory backgrounds with vanishing fluxes since the calibrated cycles are the cycles static probe branes can wrap whilst preserving supersymmetry. Generalised calibrations [56] are the natural generalisation to backgrounds when the fluxes are non-vanishing. Important work relating calibrations and the superpotential of the effective theory on  $E$  first appeared in [51]. Starting with the work [39] and subsequent work including [37], [38], [76] it has become clear that the conditions placed on supersymmetric backgrounds often have the useful physical interpretation as generalised calibrations. The reason for this is simply that the backgrounds correspond to branes wrapping calibrated cycles after taking into account the back-reaction (see [38] for further discussion). Such wrapped brane solutions were first found in [74] and a review can be found in [36]. The relationship between wrapped and intersecting brane solutions and generalised calibrations has been studied in [22], [40], [63]. The classification of supersymmetric solutions using  $G$ -structures has also led to a further exploration of generalised calibrations for non-static brane configurations [58]. Further work, specifically on the relationship between supersymmetry and generalized calibrations in flux compactifications, has appeared in [39], [38], [21].

## 2 Supersymmetry and $G$ -structures

### 2.1 Some supergravity

Let us start by characterising the type of problem we are trying to solve. First we summarise a few relevant parts of the supergravity theories which arise in string theory and then describe the notion of a supersymmetric compactification or reduction. We will concentrate on two fairly generic examples in ten and eleven dimensions.

We start with a *supergravity theory* on a  $d$ -dimensional Lorentzian spin manifold  $M$ . This is an approximation to the full string theory valid in the limit where the curvature of the manifold is small compared to the intrinsic string scale. The supergravity is described in terms a number of fields, including the *bosonic* fields

$$\begin{aligned} g & \text{ Lorentzian metric,} \\ \Phi \in C^\infty(M) & \text{ dilaton,} \\ F^{(p)} \in C^\infty(\Lambda^p T^*M) & \text{ } p\text{-form fluxes} \end{aligned} \quad (2.1)$$

for certain values of  $p$  satisfying equations of motion which are generalisations of Einstein's and Maxwell's equations. Particular  $p$ -form fluxes are also sometimes labelled  $G$  or  $H$ . For the cases we will consider, the dilaton  $\Phi$  is either not present in the theory or assumed to be zero. Since the theory is supersymmetric these fields are paired with a set of *fermionic* fields transforming in spinor representations. However, these will all be set to zero in the backgrounds we consider. A bosonic solution of the equations of motion is called a supergravity *background*.

We would like to characterise *supersymmetric backgrounds*. Let  $S \rightarrow M$  be a spin bundle. (Precisely which spinor representation we have depends on the dimension  $d$  and the type of supergravity theory.) The supergravity theory defines a particular connection

$$D: C^\infty(S) \rightarrow C^\infty(S \otimes T^*M) \quad \text{supergravity connection,} \quad (2.2)$$

in terms of the metric, dilaton and  $p$ -form fluxes. A background is supersymmetric if we have a non-trivial solution to

$$D\epsilon = 0 \quad \text{Killing spinor equation,} \quad (2.3)$$

for  $\epsilon \in C^\infty(S)$ . If we have  $n$  independent solutions then the background is said to preserve  $n$  supersymmetries. (Often the supergravity also defines a map  $P \in C^\infty(\text{End}(S))$  in terms of the dilaton and  $F^{(p)}$  and a supersymmetric solution must simultaneously satisfy the ‘‘dilatin equation’’  $P\epsilon = 0$ . For our particular examples either  $P$  is not present in the supergravity theory or is assumed to be identically zero.)

The two cases we will consider are (1)  $d = 11$  supergravity with four-form flux  $G$  and (2)  $d = 10$  *Type IIB* supergravity keeping only a self-dual five-form flux  $F^{(5)} = *F^{(5)}$ . The corresponding supergravity connections are given, in components, by

$$D = \nabla^g + \frac{1}{12} G \lrcorner \Gamma^{(5)} + \frac{1}{6} \Gamma^{(3)} \lrcorner G \quad d = 11, \quad (2.4)$$

$$D = \nabla^g \otimes \text{id} - \frac{1}{8} \Gamma^{(4)} \lrcorner F^{(5)} \otimes i\sigma_2 \quad \text{Type IIB,} \quad (2.5)$$

where  $\nabla^g$  is Levi-Civita connection for  $g$ . In the first case, the spinor  $\epsilon$  is a 32-dimensional real representation  $\Delta_{10,1}^{\mathbb{R}}$  of  $\text{Spin}(10, 1)$  while in the second case  $\epsilon$  is a *pair* of spinors  $(\epsilon_1, \epsilon_2)$  each in the 16-dimensional real, chiral representation  $\Delta_{9,1}^{+\mathbb{R}}$  of  $\text{Spin}(9, 1)$ . The gamma matrices  $\Gamma$  generate  $\text{Cliff}(d-1, 1)$  and  $\Gamma^{(p)}$  is the antisymmetrised product of  $p$  gamma matrices. The matrices  $\text{id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  and

$i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  act on the doublet of spinors  $\epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}$ . In index notation we have  $(v \lrcorner w)_{M_{p+1} \dots M_q} = \frac{1}{p!} v^{M_1 \dots M_p} w_{M_1 \dots M_p M_{p+1} \dots M_q}$ .

## 2.2 The problem

It is interesting to determine what the existence of solutions to the Killing spinor equation implies about the geometry of  $M$ , in general. For example, this has been studied in [37] for the most general solutions of supergravity in eleven dimensions. However, here we are concerned with a more restricted problem.

First we assume we have a *compactification*, where the topology of  $M$  is taken to be a product

$$M = E \times X \quad (2.6)$$

of a  $(d-n)$ -dimensional *external* manifold  $E$  and an  $n$ -dimensional *internal* manifold  $X$ . Although compact  $X$  is often of most interest, by an abuse of terminology, we will also allow for non-compact  $X$ . Next, the metric is taken to be a warped product. In particular we consider two cases

$$\begin{aligned} g &= e^{2\Delta} \eta_{d-n} + g_X && \text{flat space,} \\ g &= e^{2\lambda} \phi_{d-n} + g_X && \text{AdS space,} \end{aligned} \quad (2.7)$$

where  $g_X$  is a Riemannian metric on  $X$  while  $\eta_r$  is the flat Minkowski metric on  $E = \mathbb{R}^{r-1,1}$  and  $\phi_r$  is the constant curvature metric on anti-de Sitter space  $E = \text{AdS}_r$ . In the latter case  $\text{Ric}_{\phi_r} = -(r-1)m^2\phi_r$  where  $m$  is the inverse radius of the AdS space. In each case

$$\lambda, \Delta \in C^\infty(X). \quad (2.8)$$

Finally the dilaton and fluxes are assumed to be given by objects on  $X$ , so that in the two cases we have

$$\begin{aligned} F^{(p)} &= \begin{cases} e^{(d-n)\Delta} \text{vol}_{\eta_{d-n}} \wedge f + h, & f \in C^\infty(\Lambda^{n-d+p} T^*X), \\ e^{(d-n)\lambda} \text{vol}_{\phi_{d-n}} \wedge f + h, & h \in C^\infty(\Lambda^p T^*X), \end{cases} \\ \Phi &\in C^\infty(X). \end{aligned} \quad (2.9)$$

where  $\text{vol}_{g_E}$  is the volume form corresponding to the metric  $g_E \in \{\eta_r, \phi_r\}$  on  $E$  and  $f$  and  $h$  are sometimes referred to as the electric and magnetic fluxes.

Physically such solutions are interesting because, first, in the flat-space case with  $d-n=4$ , the space  $E$  is a model for four-dimensional particle physics. Secondly, the AdS-CFT correspondence [1] implies that such AdS geometries should be gravitational duals of conformal field theories in  $d-n-1$  dimensions. The particular internal  $X$  geometry encodes the content of the particular conformal field theory.

Given this product ansatz the Killing spinor equation (2.3) reduces to equations on a spinor  $\psi$  of  $\text{Spin}(d-n-1, 1)$  on  $E$  and a spinor, not necessarily irreducible,  $\xi$

of  $\text{Spin}(n)$  on  $X$ . The exact decomposition of  $\epsilon \in C^\infty(S)$  depends on the dimensions  $d$  and  $n$ . In all cases one takes  $\psi$  to satisfy the standard Killing spinor equation on  $E$ , that is

$$\begin{aligned} \nabla^\eta \psi &= 0 \quad \text{flat space,} \\ (\nabla^\phi - \tfrac{1}{2}m\rho) \psi &= 0 \quad \text{AdS space,} \end{aligned} \quad (2.10)$$

where  $\nabla^\eta$  and  $\nabla^\phi$  are the Levi-Civita connections for  $\eta_{d-n}$  and  $\phi_{d-n}$  respectively and  $\rho$  are gamma matrices for  $\text{Spin}(d-n-1, 1)$ . If  $S^X \rightarrow X$  is the spin bundle on  $X$  coming from the decomposition of  $S$ , the Killing spinor equation (2.3) then has the form<sup>1</sup>

$$D^X \xi = 0, \quad Q^X \xi = 0, \quad \text{reduced Killing spinor eqns.,} \quad (2.11)$$

where the connection  $D^X: C^\infty(S^X) \rightarrow C^\infty(S^X \otimes T^*X)$  and the map  $Q^X: C^\infty(S^X) \rightarrow C^\infty(S^X)$  each are defined in terms of flux, dilaton, and  $\Delta$  or  $\lambda$  and  $m$ . The condition  $D^X \xi = 0$  comes from the reduction of  $D\epsilon = 0$  on  $X$  and  $Q^X \xi = 0$  from the reduction on  $E$ .

Our basic question is then:

*What does the existence of solutions to the reduced Killing spinor equations imply about the geometry of  $X$  and the form of the fluxes and dilaton?*

In general we want to translate the Killing spinor conditions into some convenient set of necessary and sufficient conditions, such as, for instance,  $X$  has a particular almost complex or contact structure or a particular Killing vector.

Let us end with a couple of further comments. First note that there is a connection between the two types of compactification (2.7). Consider  $E \times X = \text{AdS}_{d-n} \times X$ . Locally we can write the AdS metric  $\phi_{d-n}$  in Poincaré coordinates

$$\phi_{d-n} = e^{-2mr} \eta_{d-n-1} + dr \otimes dr. \quad (2.12)$$

Thus we have

$$\begin{aligned} g &= e^{2\lambda} \phi_{d-n} + g_X = e^{2\lambda} e^{-2mr} \eta_{d-n-1} + (e^{2\lambda} dr \otimes dr + g_X) \\ &\equiv e^{2\Delta} \eta_{d-n} + g_{X'}, \end{aligned} \quad (2.13)$$

where

$$\begin{aligned} \Delta &= \lambda - mr, \\ g_{X'} &= e^{2\lambda} dr \otimes dr + g_X. \end{aligned} \quad (2.14)$$

and hence an AdS compactification on  $X$  to  $\text{AdS}_{d-n}$  is really a special case of a flat space compactification on  $X' = X \times \mathbb{R}^+$  to  $\mathbb{R}^{d-n-2,1}$ . This will be particularly useful for deriving the conditions on the geometry of AdS compactifications in what follows.

Next, recall that to be a true background the fields also have to satisfy the supergravity equations of motion. Part of these are a set of Bianchi identities involving the

---

<sup>1</sup>If there was also originally a  $P\epsilon = 0$  condition this also reduces to a further condition  $P^X \xi = 0$  with  $P^X \in C^\infty(\text{End}(S^X))$ .



exterior derivatives of  $F^{(p)}$ . In general one can derive equations involving the Ricci tensor and derivatives of the fluxes by considering integrability conditions, such as  $D^2\epsilon = 0$ , for the Killing spinor equations. One can show, following [27], [37], that, for product backgrounds of the form (2.7), once one imposes the Bianchi identities and the equation of motion for the flux, the other equations of motion follow from these integrability conditions. In fact, for the cases we consider, the flux equation of motion is also implied by the supersymmetry conditions and so if we have a solution of the Killing spinor equation (2.3) and in addition the Bianchi identity

$$dG = 0 \quad \text{or} \quad dF^{(5)} = 0, \quad (2.15)$$

then we have a solution of the equations of motion. When  $E = \text{AdS}$ , at least for the cases considered here, the supersymmetry conditions are even stronger: any solution of the Killing spinor equations is necessarily a solution of the equations of motion [42].

To have truly a string or M-theory background as opposed to a supergravity solution there is also a “quantisation” condition on the fluxes. For  $n < 8$  the equations of motion for  $G$  gives  $d * G = 0$  while  $d * F^{(5)} = 0$  is implied by the Bianchi identity since  $F^{(5)}$  is self-dual. Hence in both cases the fluxes are harmonic. To be a true string or M-theory background, we have the quantisation condition  $G \in H^4(X, \mathbb{Z})$  or  $F^{(5)} \in H^5(X, \mathbb{Z})$ . More precisely the fluxes represent classes in K-theory [79]. In the AdS-CFT correspondence, these integer classes are related to integral parameters in the field theory.

### 2.3 $G$ -structures

Our approach for analysing what solutions to the reduced Killing spinor equations (2.11) imply about the geometry of  $X$  will use the language of  $G$ -structures. Let us start with a brief review. For more information see for instance [69] or [83].

Let  $F$  be the frame bundle of  $X$ , then

a  $G$ -structure is a principle sub-bundle  $P$  of  $F$  with fibre  $G \subset \text{GL}(n, \mathbb{R})$ .

For example if  $G = \text{O}(n)$ , the sub-bundle is interpreted as the set of orthonormal frames and defines a metric. Let  $\nabla$  be a connection on  $F$  or equivalently the corresponding connection on  $TM$ . One finds

- (1) given a  $G$ -structure, all tensors on  $X$  can be decomposed into  $G$  representations;
- (2) if  $\nabla$  is *compatible* with the  $G$ -structure, that is, it reduces to a connection on  $P$ , then  $\text{Hol}(TX, \nabla) \subseteq G$ ;
- (3) there is an *obstruction* to finding torsion-free compatible  $\nabla$ , measured by the *intrinsic torsion*  $T_0(P)$ , which can be used to classify  $G$ -structures.

The intrinsic torsion is defined as follows. Given a pair  $(\nabla', \nabla)$  of compatible connections, viewed as connections on  $P$  we have  $\nabla' - \nabla \in C^\infty(\text{ad } P \otimes T^*X)$ . Let  $T(\nabla) \in C^\infty(TX \otimes \Lambda^2 T^*X)$  be the torsion of  $\nabla$ . We can then define a map

$\sigma_P : C^\infty(\text{ad } P \otimes T^*X) \rightarrow C^\infty(TX \otimes \Lambda^2 T^*X)$  given by

$$\alpha = \nabla' - \nabla \mapsto \sigma_P(\alpha) = T(\nabla') - T(\nabla), \quad (2.16)$$

and hence we have the quotient bundle  $\text{Coker } \sigma_P = TX \otimes \Lambda^2 T^*X / \text{ad } P \otimes T^*X$ . Let the intrinsic torsion  $T_0(P)$  be the image of  $T(\nabla)$  in  $\text{Coker } \sigma_P$  for any compatible connection  $\nabla$ . By definition it is the part of the torsion independent of the choice of compatible connection and only depends on the  $G$ -structure  $P$ .

We will be interested in the particular class of  $G$ -structures where

- (1)  $P$  can be defined in terms of a finite set  $\eta$  of  $G$ -invariant tensors on  $X$ ,
- (2)  $G \subset \text{O}(n)$ .

Prime examples of the former condition are an almost complex structure with  $G = \text{GL}(k, \mathbb{C}) \subset \text{GL}(2k, \mathbb{R})$ , or an  $\text{O}(n)$ -structure defined by a metric  $g$ . The sub-bundle of frames  $P$  is defined by requiring the tensors to have a particular form. For instance, for the  $\text{O}(n)$ -structure we define  $P$  as the set of frames such that the metric  $g$  has the form

$$g = e^1 \otimes e^1 + \cdots + e^n \otimes e^n. \quad (2.17)$$

These restrictions imply a number of useful results. From the first condition it follows that

$$\nabla \text{ is compatible with } P \Leftrightarrow \nabla \Xi = 0 \quad \text{for all } \Xi \in \eta. \quad (2.18)$$

The second condition implies that  $P$  defines a metric  $g$  and hence an  $\text{O}(n)$  structure  $Q$ . A key point, given  $\text{ad } Q \cong \Lambda^2 T^*X$ , is that  $\sigma_Q$  is in fact an isomorphism and hence

an  $\text{O}(n)$ -structure with metric  $g$  has a unique compatible torsion-free connection, namely the Levi-Civita connection  $\nabla^g$ .

Any  $P$ -compatible connection  $\nabla$  can then be written as  $\nabla = \nabla^g + \alpha + \alpha^\perp$  where  $\alpha$  is a section of  $\text{ad } P \otimes T^*X$  while  $\alpha^\perp$  is a section of  $(\text{ad } P)^\perp \otimes T^*X$  with  $(\text{ad } P)^\perp = \text{ad } Q / \text{ad } P$ . Furthermore  $\text{Coker } \sigma_P \cong (\text{ad } P)^\perp \otimes T^*X$  and given the isomorphism  $\sigma_Q$ , we see that  $T_0(P)$  can be identified with  $\alpha^\perp$ . Equivalently, since by definition  $\nabla \Xi = (\nabla^g + \alpha^\perp) \Xi = 0$  for any  $\Xi \in \eta$ , we have

$$T_0(P) \text{ can be identified with the set } \{\nabla^g \Xi : \Xi \in \eta\}. \quad (2.19)$$

Finally, if  $T_0(P) = 0$  then  $\nabla^g$  is compatible with  $P$  and  $X$  has *special holonomy*, that is, for  $G \subset \text{O}(n)$

$$T_0(P) = 0 \Leftrightarrow \text{Hol}(X) \subseteq G, \quad (2.20)$$

where  $\text{Hol}(X) \equiv \text{Hol}(TX, \nabla^g)$ .

A number of examples of such  $G$ -structures, familiar from the discussion of special holonomy manifolds, are listed in Table 1. Except for  $g$  in  $\text{Spin}(7)$  all the elements of  $\eta$  are forms, where, in the table, the subscript denotes the degree. Consider for instance the case  $G = \text{SU}(k)$  in dimension  $n = 2k$ . This includes Calabi–Yau  $k$ -folds in the special case that  $T_0(P) = 0$ . The elements of  $\eta$  are the fundamental two-form

$J$  and the complex  $k$ -form  $\Omega$ . The structure  $P$  is defined as the set of frames where  $J$  and  $\Omega$  have the form

$$\begin{aligned} J &= e^1 \wedge e^2 + \cdots + e^{n-1} \wedge e^n, \\ \Omega &= (e^1 + ie^2) \wedge \cdots \wedge (e^{n-1} + ie^n). \end{aligned} \quad (2.21)$$

The two-form  $J$  is invariant under  $\mathrm{Sp}(k, \mathbb{R}) \subset \mathrm{GL}(2k, \mathbb{R})$  and  $\Omega$  is invariant under  $\mathrm{SL}(k, \mathbb{C}) \subset \mathrm{GL}(2k, \mathbb{R})$ . The common subgroup is  $\mathrm{SU}(k) \subset \mathrm{SO}(2k)$ . Thus the pair  $J$  and  $\Omega$  determines a metric. For  $\mathrm{SU}(k)$ -holonomy we then require that the intrinsic torsion vanishes or equivalently  $\nabla^g J = \nabla^g \Omega = 0$  and  $J$  is then the Kähler form and  $\Omega$  the holomorphic  $k$ -form. By considering the corresponding  $\mathrm{SU}(k)$ -representations, it is easy to show [38] that

$$T_0(P) \text{ can be identified with the set } \{dJ, d\Omega\}, \quad (2.22)$$

so that  $\mathrm{Hol}(X) \subseteq \mathrm{SU}(k)$  is equivalent to  $\{dJ = 0, d\Omega = 0\}$ . This result that  $T_0(P)$  is encoded in the exterior derivatives  $d\Xi$  for  $\Xi \in \eta$  is characteristic of all the examples in Table 1.

Table 1.  $G$ -structures and supersymmetry.

| dimension | special holo. space $X$ | $G \subset \mathrm{SO}(n)$ | $\eta$                                | no. of supersyms.    |
|-----------|-------------------------|----------------------------|---------------------------------------|----------------------|
| $n = 2k$  | Calabi–Yau              | $\mathrm{SU}(k)$           | $\{J_2, \Omega_k\}$                   | $d_\epsilon/2^{k-1}$ |
| $n = 4k$  | hyper-Kähler            | $\mathrm{Sp}(k)$           | $\{J_2^{(1)}, J_2^{(2)}, J_2^{(3)}\}$ | $d_\epsilon/2^k$     |
| $n = 7$   | $G_2$                   | $G_2$                      | $\{\phi_3\}$                          | $d_\epsilon/8$       |
| $n = 8$   | $\mathrm{Spin}(7)$      | $\mathrm{Spin}(7)$         | $\{g, \Psi_4\}$                       | $d_\epsilon/16$      |

## 2.4 Supersymmetry and $G$ -structures

We can now use the language of  $G$ -structures to characterise the constraints on the geometry of  $X$  due to the existence of solutions to the Killing spinor equations (2.11). We define the space of solutions

$$C = \{\xi \in C^\infty(S^X) : D^X \xi = 0, Q^X \xi = 0\}, \quad (2.23)$$

which defines a sub-bundle of  $S^X$ . The basic idea is that the existence of  $C$  implies that there is a sub-bundle  $P$  of the frame bundle and hence a  $G$ -structure.

First note that since we have spinors we have an  $\mathrm{SO}(n)$ -structure  $Q$  defined by the metric and orientation and a spin structure  $(\tilde{Q}, \pi)$ , where  $\tilde{Q}$  is a  $\mathrm{Spin}(n)$  principle bundle and  $\pi : \tilde{Q} \rightarrow Q$  is the covering map modelled on the double cover  $\mathrm{Spin}(n) \rightarrow \mathrm{SO}(n)$ . For any  $n$  the Clifford algebra  $\mathrm{Cliff}(n)$  is equivalent to a general linear group acting on the vector space of spinors  $\xi$  and implying we can also define a  $\mathrm{Cliff}(n)$

principle bundle  $\tilde{\Gamma}$  with  $\tilde{Q} \subset \tilde{\Gamma}$ . Recall that  $D^X$  is a Clifford connection defined on  $\tilde{\Gamma}$  and generically does not descend to a connection on  $\tilde{Q}$ .

Let  $K_x \subset \text{Cliff}(n)$  be the stabilizer group in the Clifford algebra of  $C|_x$ , the set of solutions  $C$  evaluated at the point  $x \in X$ . Since  $D^X$  is a  $\text{Cliff}(n)$  connection, by parallel transport  $K = K_x$  is independent of  $x \in X$ , and hence  $C$  defines a  $K$  principle sub-bundle  $\tilde{\Lambda} \subset \tilde{\Gamma}$  built from those elements of  $\tilde{\Gamma}$  leaving  $C$  invariant. We can equally well consider the stabilizer  $\tilde{G}_x \subset \text{Spin}(n)$  of  $C|_x$  in the spin group. Since  $D^X$  does not descend to  $\tilde{Q}$  in general  $\tilde{G}_x$  is not independent of  $x \in X$  and hence the stabilizer does not define a sub-bundle of  $\tilde{Q}$ . However, since there is only a finite number of possible stabiliser groups, we can still define a unique  $\tilde{G} = \tilde{G}_x$  with  $x \in U$  for some open subset of  $U \subset X$  (with possibly non-trivial topology). Or alternatively we can restrict our considerations to  $C$  such that  $\tilde{G}$  is globally defined. In this way  $C$  defines a sub-bundle  $\tilde{P} \subset \tilde{Q}$  of the spin bundle. The double cover  $\pi$  then restricts to a projection  $\pi: \tilde{P} \rightarrow P \subset Q$  and hence we have a  $G$ -structure  $P$  where  $G$  is the projection of  $\tilde{G}$ . (In fact in all cases we consider  $\tilde{G} = G$  and  $P \cong \tilde{P}$ .) In conclusion we see that

- (1)  $C$  defines a  $G$ -structure  $P$  over (at least) some open subset  $U \subset X$  where  $G \subset \text{SO}(n)$ .

The different structures and groups can be summarized as follows:

$$\begin{array}{ccc}
 \tilde{\Gamma} & \longleftarrow & \tilde{\Lambda} & \text{Cliff}(n) & \longleftarrow & K \\
 \uparrow & & \uparrow & \uparrow & & \uparrow \\
 \tilde{Q} & \longleftarrow & \tilde{P} & \text{Spin}(n) & \longleftarrow & \tilde{G} \\
 \pi \downarrow & & \pi \downarrow & \pi \downarrow & & \pi \downarrow \\
 Q & \longleftarrow & P & \text{SO}(n) & \longleftarrow & G
 \end{array} \tag{2.24}$$

Note we can equivalently think of defining  $\tilde{P}$  as the intersection  $\tilde{\Lambda} \cap \tilde{Q}$  as embeddings in  $\tilde{\Gamma}$ . Generically this is not a bundle defined over the whole of  $X$  since the fibre group can change, reflecting the fact that  $P$  is generically only defined over  $U \subset X$ . Note also that, by construction,

$$\text{Hol}(S^X, D^X) \subseteq K. \tag{2.25}$$

This corresponds to the notion of generalised holonomy introduced by Duff and Liu [28]. Note, however, that this misses the important information that there is also a spin structure  $\tilde{Q} \subset \tilde{\Gamma}$  in the Clifford bundle. In other words the full information is contained in the pair  $(\tilde{\Lambda}, \tilde{Q})$ , which at least in a patch  $U$  translates into the  $G$ -structure  $P$ .

To see explicitly that  $C$  defines a  $G$ -structure recall that the Clifford algebra gives us a set of maps  $w_p: C^\infty(S^X \otimes S^X) \rightarrow C^\infty(\Lambda^p TX)$  given by

$$(\xi, \chi) \mapsto w_p(\xi, \chi) = \bar{\xi} \gamma^{(p)} \chi, \tag{2.26}$$

where  $\gamma^{(p)}$  is the antisymmetric product of  $p$  gamma matrices generating the Clifford algebra  $\text{Cliff}(n)$ . By construction if  $\xi, \chi \in C$  then  $w_p(\xi, \chi)$  is invariant under  $G$ . The invariant forms  $\Xi \in \eta$  defining  $P$  are then generically constructed from combinations of bilinears of the form  $w_p(\xi, \chi)$ . Specific examples will be given in the next section.

Finally, since  $D^X$  is determined by the flux, dilaton, and  $\Delta$  or  $\lambda$  and  $m$ , from the discussion of the last section, we have our second result

- (2) the intrinsic torsion  $T_0(P)$  is determined in terms of the flux, dilaton, and  $\Delta$  or  $\lambda$  and  $m$ .

Generically, however, there may be components of, for instance, the flux which are not related to  $T_0(P)$ . Thus we see that the existence of solutions to the Killing spinor equations (2.11) translates into the existence of a  $G$ -structure  $P$  with specific intrinsic torsion  $T_0(P)$ .

As mentioned above, in some cases the  $G$ -structure is globally defined. On the other hand, in some cases the  $G$ -structure is only defined locally in some open set, and possibly only in a topologically trivial neighbourhood. Of course in such a neighbourhood the structure group of the frame bundle can always be reduced to the identity structure. However, the key point is that supersymmetry defines a canonical  $G$ -structure that can be used to give a precise characterisation of the local geometry of the solution. In turn, as we shall see, this often provides a powerful method to construct explicit local supersymmetric solutions. Furthermore, the global properties of such solutions can then be found by determining the maximal analytic extension of the local solution (this is a standard technique used in the physics literature).

Let us now see how this description in terms of  $G$ -structures works in a couple of specific examples relevant to the new solutions we will discuss later.

**Example 1.  $n = 6$  in Type IIB.** Consider the case of type IIB supergravity with  $M = \mathbb{R}^{3,1} \times X$  and only the self-dual five-form non-vanishing (the most general case is considered in [24]). First we need the spinor decomposition. Recall that  $\epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}$  is a section of  $S = S_+ \oplus S_+$  where the spin bundle  $S_+$  corresponds to the real (positive) chirality spinor representation  $\Delta_{9,1}^{+\mathbb{R}}$  of  $\text{Spin}(9, 1)$ . In general we have that the complexified representation decomposes as

$$(\Delta_{9,1}^{+\mathbb{R}})_\mathbb{C} = \Delta_{3,1}^+ \otimes \Delta_6^+ + \overline{\Delta_{3,1}^+} \otimes \overline{\Delta_6^+}, \quad (2.27)$$

where  $\Delta_{3,1}^+$  and  $\Delta_6^+$  are the complex positive chirality representations of  $\text{Spin}(3, 1)$  and  $\text{Spin}(6)$ . The bar denotes the conjugate representation. Let  $S_{3,1}^+$  and  $S_6^+$  be the corresponding spin bundles. To ensure that the  $\epsilon_i$  are real we decompose

$$\epsilon_i = \psi \otimes e^{\Delta/2} \xi_i + \psi^c \otimes e^{\Delta/2} \xi_i^c, \quad (2.28)$$

where  $\psi \in C^\infty(S_{3,1}^+)$ ,  $\xi_i \in C^\infty(S_6^+)$  while  $\psi^c$  and  $\xi_i^c$  are the complex conjugate spinors and we have included factors of  $e^{\Delta/2}$  in the definition of  $\xi_i$  for convenience. We then define the combinations  $\xi^\pm = \xi_1 \pm i\xi_2$ .

The self-dual five-form flux ansatz (2.9) can be written as

$$F^{(5)} = e^{4\Delta} \text{vol}_{\eta_4} \wedge f - *_X f, \quad f \in C^\infty(\Lambda^1 T^*X), \quad (2.29)$$

where  $*_X$  is the Hodge star defined using  $g_X$  on  $X$ . Decomposing the Killing spinor equations it is easy to show that either  $\xi^+ = 0$  or  $\xi^- = 0$ . Let us assume that  $\xi^- = 0$  then, defining  $\xi = \xi^+$  with  $S^X = S_6^+$ , we have

$$\begin{aligned} D^X &= \nabla^{g_X} + \frac{1}{8} f \lrcorner \gamma^{(2)}, \\ Q^X &= \gamma^{(1)} \lrcorner d\Delta + \frac{1}{4} \gamma^{(1)} \lrcorner f. \end{aligned} \quad (2.30)$$

Note that  $D^X$  involves only  $\gamma^{(2)}$  and so in this case it does descend to a metric compatible connection  $\nabla$  on  $TX$ . Thus, in this case, the  $G$ -structure to be discussed next, is in fact globally defined.

We will consider the case of the minimum number of preserved supersymmetries where the set of solutions  $C$  is one-dimensional, corresponding to non-zero multiples of some fixed solution  $\xi \in C$ . The stabiliser of a single spinor is  $SU(3)$  and thus we have

$$X \text{ has } SU(3)\text{-structure.} \quad (2.31)$$

It is easy to show that  $\nabla^{g_X}(\bar{\xi}\xi) = 0$ . If we choose to normalise such that  $\bar{\xi}\xi = 1$ , it then follows that the elements of  $\eta$  fixing the  $SU(3)$  structure are given by the bilinears

$$J = -i\bar{\xi}\gamma^{(2)}\xi, \quad \Omega = \bar{\xi}^c\gamma^{(3)}\xi. \quad (2.32)$$

We next calculate the intrinsic torsion. Recall that this is contained in  $dJ$  and  $d\Omega$ . From (2.30) one finds

$$\begin{aligned} d(e^{4\Delta}) &= -e^{4\Delta} f, \\ d(e^{2\Delta} J) &= 0, \\ d(e^{3\Delta} \Omega) &= 0, \end{aligned} \quad (2.33)$$

which completely determines the intrinsic torsion, as well as the flux, in terms of  $d\Delta$ . This implies that  $g_X$  is conformally Calabi–Yau, that is we can write

$$g_X = e^{-2\Delta} g_6, \quad (2.34)$$

where  $g_6$  has integrable  $SU(3)$ -structure. In addition

$$f = -4d\Delta. \quad (2.35)$$

Note that the Bianchi identity for  $F^{(5)}$  is satisfied provided we have  $d *_X f = 0$ . This translates in to the harmonic condition

$$\nabla_{g_6}^2 e^{-4\Delta} = 0, \quad (2.36)$$

where  $\nabla_{g_6}^2$  is the Laplacian for  $g_6$  on  $X$ . Thus we have completely translated the conditions for a supersymmetric background into a geometrical constraint (2.34) together with a solution of the Laplacian (2.36).

**Example 2.  $n = 7$  in  $d = 11$ .** Now consider the case of  $d = 11$  supergravity on  $M = \mathbb{R}^{3,1} \times X$ . (Here we are following the discussion of [66].) Again we start with the spinor decomposition. Recall that the  $d = 11$  spinor  $\epsilon$  is a section of a spin bundle corresponding to the real 32-dimensional representation  $\Delta_{10,1}^{\mathbb{R}}$  of  $\text{Spin}(10, 1)$ . Under  $\text{Spin}(3, 1) \times \text{Spin}(7)$  the complexified representation decomposes as

$$(\Delta_{10,1}^{\mathbb{R}})_{\mathbb{C}} = \Delta_{3,1}^+ \otimes (\Delta_7^{\mathbb{R}})_{\mathbb{C}} + \overline{\Delta_{3,1}^+} \otimes (\Delta_7^{\mathbb{R}})_{\mathbb{C}}, \quad (2.37)$$

where  $\Delta_{3,1}^+$  and  $\Delta_7^{\mathbb{R}}$  are the complex positive chirality representation of  $\text{Spin}(3, 1)$  and real representation of  $\text{Spin}(7)$  respectively. Let  $S_{3,1}^+$  and  $S_7^{\mathbb{R}}$  be the corresponding spin bundles. To ensure that the  $\epsilon_i$  are real we decompose

$$\epsilon = \psi \otimes (\xi_1 + i\xi_2) + \psi^c \otimes (\xi_1 - i\xi_2), \quad (2.38)$$

where  $\psi \in C^\infty(S_{3,1}^+)$ ,  $\xi_i \in C^\infty(S_7^{\mathbb{R}})$ .

In the flux ansatz (2.9) we assume  $G$  is pure magnetic so

$$G \in C^\infty(\Lambda^4 T^*X). \quad (2.39)$$

Defining  $\xi$  as the doublet  $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$  with  $S^X = S_7^{\mathbb{R}} \oplus S_7^{\mathbb{R}}$ , we find

$$\begin{aligned} D^X &= \nabla^{g_X} \otimes \text{id} + \frac{1}{12} \gamma^{(2)} \lrcorner^* X G \otimes i\sigma_2 + \frac{1}{6} i\gamma^{(3)} \lrcorner G \otimes i\sigma_2, \\ Q^X &= \gamma^{(1)} \lrcorner d\Delta \otimes \text{id} + \frac{1}{6} i\gamma^{(4)} \lrcorner G \otimes i\sigma_2, \end{aligned} \quad (2.40)$$

where  $i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . Note that  $D^X$  does not descend to a metric compatible connection  $\nabla$  on  $TX$ .

Again we are interested in the minimum number of preserved supersymmetries so the set of solutions  $C$  is one-dimensional, corresponding to non-zero multiples of some fixed solution  $\xi \in C$ . In addition we will assume the  $\xi_i$  in  $\xi$  are each non-zero and more importantly

$$\bar{\xi}_1 \xi_2 = 0. \quad (2.41)$$

(Note that the generic conditions, without this assumption, were derived in [72].) It is then easy to show that the  $e^{-\Delta/2 \xi_i}$  have constant norm. Together with (2.41) this then implies that the stabiliser of  $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$  is  $G = \text{SU}(3)$  independent of  $x \in X$  and hence

$$X \text{ has } \text{SU}(3)\text{-structure}. \quad (2.42)$$

Note that this is an  $\text{SU}(3)$ -structure in seven dimensions.

If we normalise  $\xi \in C$  such that  $e^{-\Delta} \bar{\xi}_1 \xi_1 = e^{-\Delta} \bar{\xi}_2 \xi_2 = 1$  then the elements of  $\eta = \{J, \Omega, K\}$  fixing the  $SU(3)$  structure are given by

$$\begin{aligned} J &= -e^{-\Delta} \bar{\xi}_1 \gamma^{(2)} \xi_2, \\ \Omega &= -\frac{1}{2} e^{-\Delta} (\bar{\xi}_1 \gamma^{(3)} \xi_1 - \bar{\xi}_2 \gamma^{(3)} \xi_2) + i e^{-\Delta} \bar{\xi}_1 \gamma^{(3)} \xi_2, \\ K &= -i e^{-\Delta} \bar{\xi}_1 \gamma^{(1)} \xi_2, \end{aligned} \quad (2.43)$$

where we are using the convention  $i\gamma^{(7)} = \text{vol}_X \text{ id}$ . The one-form  $K$  defines a product structure  $R \subset Q$  with fibre  $SO(6) \subset SO(7)$  and then  $J$  and  $\Omega$  define the  $G$ -structure  $P \subset R$  with fibre  $SU(3)$ .

As in six dimensions the intrinsic torsion of an  $SU(3)$ -structure in seven dimensions is completely determined by the exterior derivatives of  $K$ ,  $J$  and  $\Omega$ . One finds

$$\begin{aligned} d(e^{2\Delta} K) &= 0, \\ d(e^{4\Delta} J) &= e^{-4\Delta} *_X G, \\ d(e^{3\Delta} \Omega) &= 0, \\ d(e^{2\Delta} J \wedge J) &= -2 e^{2\Delta} G \wedge K. \end{aligned} \quad (2.44)$$

These equations were derived in [66] (though with a different factor in the last equation, as discussed in [42]). It was argued in [25] that these are the necessary and sufficient conditions for a geometry to admit a single Killing spinor. Furthermore, the second equation implies the  $G$  equation of motion and thus, given an integrability argument as in [37], only the Bianchi identity  $dG$  need be imposed to give a solution to the full equations of motion.

## 2.5 Relation to generalised calibrations

It turns out that there is a very interesting relation between the torsion conditions, such as (2.33) and (2.44), one derives for supersymmetric backgrounds and the notion of a “generalised calibrations” introduced in [56]. This gives a very physical interpretation of the conditions in terms of string theory “branes”. Here we will only briefly touch on this relation.

Let us start by recalling the notion of calibrations and a calibrated cycle [59], [23] (for a review see [69], [36]) Suppose we have a Riemannian manifold  $X$  with metric  $g_X$  and let  $\xi \subset T_x X$  be an oriented  $p$ -dimensional tangent plane at any point  $x \in X$ . We can then define  $\text{vol}_\xi$  as the volume form on  $\xi$  built from the restriction  $g_X|_\xi$  of the metric to  $\xi$ . A  $p$ -form  $\Xi$  is then a *calibration* if

$$\begin{aligned} \text{(i)} \quad \Xi|_\xi &\leq \text{vol}_\xi \quad \text{for all } \xi, \\ \text{(ii)} \quad d\Xi &= 0. \end{aligned} \quad (2.45)$$



Furthermore, given a  $p$ -dimensional oriented submanifold  $C_p$ , we say  $C_p$  is *calibrated* if

$$\Xi|_{T_x C_p} = \text{vol}_{T_x C_p} \quad \text{for all } x \in C_p. \quad (2.46)$$

If  $C'_p$  is another submanifold in the same homology class we have

$$\int_{C_p} \text{vol}_{C_p} = \int_{C_p} \Xi|_{T C_p} = \int_{C'_p} \Xi|_{T C'_p} \leq \int_{C'_p} \text{vol}_{C'_p}, \quad (2.47)$$

and we get the main result that a calibrated submanifold has minimum volume in its homology class.

Now suppose we have a set of invariant tensors  $\eta$  defining a  $G$ -structure  $P$  with  $G \subset \text{SO}(n)$ . One finds that, for  $p$ -forms  $\Xi$

$$\begin{aligned} \Xi \in \eta &\Rightarrow \text{calibration condition (i),} \\ T_0(P) = 0 &\Rightarrow \text{calibration condition (ii).} \end{aligned} \quad (2.48)$$

Thus the vanishing intrinsic torsion of the  $G$ -structure (i.e. special holonomy  $G$ ) corresponds to the closure of the calibration forms. It is natural, then, to try to interpret our intrinsic torsion conditions (2.33) and (2.44) as defining “generalised calibrations” [56]. Obviously calibrated sub-manifolds will no longer be volume minimising, but one might ask if there is some more general notion of the “energy” of the submanifold which is minimised by calibrated sub-manifolds.

String theory provides precisely such an interpretation. It contains a number of extended  $p + 1$ -dimensional objects which embed into the spacetime and are known as “ $p$ -branes”: a simple example is the two-dimensional string itself. Each brane has a particular energy functional depending both on the volume of the embedded submanifold and crucially the flux and dilaton. Differential conditions such as (2.33) and (2.44) then imply that the corresponding brane energy is minimised when the submanifold is calibrated by a generalised calibration.

Consider for instance our  $d = 11$  example with  $M = \mathbb{R}^{3,1} \times X$ . The relevant branes in eleven dimensions are the “M2-brane” and the “M5-brane” and are described by embeddings of the worldvolumes  $\Sigma \hookrightarrow M$ . In particular, we can take  $\Sigma = \mathbb{R}^{r,1} \times C_s$  with  $r + s \in \{2, 5\}$ , where  $C_p$  is a  $p$ -dimensional submanifold of  $X$ . One then says that the brane is “wrapped” on  $C_s$ . Each of the conditions (2.44) can then be interpreted as generalised calibrations for different types of wrapped brane. We have

$$\begin{aligned} d(e^{2\Delta} K) &= 0 && \text{M2-brane on } C_1, \\ d(e^{4\Delta} J) &= e^{-4\Delta} *_X G && \text{M5-brane on } C_2, \\ d(e^{3\Delta} \Omega) &= 0 && \text{M5-brane on } C_3, \\ d(e^{2\Delta} J \wedge J) &= -2e^{2\Delta} G \wedge K && \text{M5-brane on } C_4. \end{aligned} \quad (2.49)$$

Note that the power of  $e^\Delta$  appearing in each expression counts the  $q + 1$  unwrapped dimensions of the brane. Roughly, the fluxes appearing on the right hand side can be

understood by noting that M2-branes couple to electric  $G$ -flux, that M5-branes couple to magnetic  $G$ -flux and that we have only kept certain components of  $G$  in our ansatz (for example the electric  $G$ -flux vanishes). The flux appearing in the last expression in (2.49) arises from the fact that there can be induced M2-brane charge on the M5-brane. For more on this correspondence see [39], [37], [38], [76].

### 3 New AdS<sub>5</sub> solutions in M-theory

We now turn to the specific problem of finding, first, the generic structure of the minimal supersymmetric configurations of  $D = 11$  supergravity with  $M = \text{AdS}_5 \times X$  and, second, a class of particular solutions of this form. Such backgrounds are of particular interest because, via the AdS-CFT correspondence, they are dual to  $\mathcal{N} = 1$  superconformal field theories in four-dimensions. This work was first presented in [42].

#### 3.1 General differential conditions

As we have seen  $\text{AdS}_5 \times X$  geometries are special cases of  $\mathbb{R}^{3,1} \times X'$  where  $X' = X \times \mathbb{R}$  with metrics and warp factors related as in eqns. (2.14). Thus we can actually obtain the general conditions for supersymmetric  $\text{AdS}_5$  compactifications from the corresponding  $n = 7$  SU(3) conditions given in eq. (2.44). (One might be concerned that these latter conditions are not completely generic, nonetheless one can show [42] that they give the generic conditions for  $\text{AdS}_5$ .)

To derive the conditions explicitly, let us denote the SU(3) structure on  $X'$  by the primed forms  $(K', J', \Omega')$ . The radial unit one-form  $e^\lambda dr$  is generically not parallel to  $K'$ ; instead we can write

$$e^\lambda dr = -\sin \zeta K' - \cos \zeta W', \quad (3.1)$$

where  $W'$  is a unit one-form orthogonal to  $K'$ . We can then define two other unit mutually orthogonal one-forms

$$\begin{aligned} K^1 &= \cos \zeta K' - \sin \zeta W' \\ K^2 &= V' = J' \cdot W' \end{aligned} \quad (3.2)$$

where  $K^1$  is the orthogonal linear combination of  $K'$  and  $W'$  and  $K^2$  is defined using  $J'^b_a$  the almost complex structure on  $X'$ . We can then define real and imaginary two-forms from the parts of  $J$  and  $\Omega$  orthogonal to  $W'$  and  $V'$ , that is

$$\begin{aligned} J &= J' - W' \wedge V' \\ \Omega &= i_{W'+iV'} \Omega' . \end{aligned} \quad (3.3)$$

Note that  $\Omega$  is not strictly a two-form on  $X$  but is a section of  $\Lambda^2 T^*X$  twisted by the complex line bundle defined by  $W' + iV'$ . This implies that the set  $(K^1, K^2, J, \Omega)$  actually defines a local  $U(2)$  structure on the six-dimensional manifold  $X$ , rather than an  $SU(2)$  structure as would be the case if  $\Omega$  were truly a two-form. Note that the structure is only local since, in particular, it breaks down when  $K'$  is parallel to  $dr$ , that is  $\cos \zeta = 0$ , in which case we cannot define  $K^1$  and  $K^2$ . Using these definitions the constraints (2.44) become

$$d(e^{3\lambda} \sin \zeta) = 2m e^{2\lambda} \cos \zeta K^1, \quad (3.4)$$

$$d(e^{4\lambda} \cos \zeta \Omega) = 3m e^{3\lambda} \Omega \wedge (-\sin \zeta K^1 + i K^2), \quad (3.5)$$

$$d(e^{5\lambda} \cos \zeta K^2) = e^{5\lambda} * G + 4m e^{4\lambda} (J - \sin \zeta K^1 \wedge K^2), \quad (3.6)$$

$$d(e^{3\lambda} \cos \zeta J \wedge K^2) = e^{3\lambda} \sin \zeta G + m e^{2\lambda} (J \wedge J - 2 \sin \zeta J \wedge K^1 \wedge K^2). \quad (3.7)$$

(Note that the  $SU(2)$  structure here differs from that used in [42] by a conformal rescaling.)

To ensure we have a solution of the equations of motion, in general one also needs to impose the equation of motion and Bianchi identity for  $G$ . The connection with the  $n = 7$  results gives us a quick way of seeing that, in fact, provided  $\sin \zeta$  is not identically zero, both conditions are a consequence of the supersymmetry constraints (3.4)–(3.7). As already noted, the equation of motion for  $G$  follows directly from the exterior derivative of the second equation in (2.44). For the Bianchi identity one notes that, given the ansatz for the  $n = 7$  metric and  $G$ , the first and last equations in (2.44) imply in general that

$$\sin \zeta dG \wedge dr = 0 \quad (3.8)$$

since  $dG$  lies solely in  $X$ . This implies that  $dG = 0$  provided  $\sin \zeta$  is not identically zero – which can only occur only when  $m = 0$  (from (3.4)). Thus we see that the constraints (3.4)–(3.7) are necessary and sufficient both for supersymmetry and for a solution of the equations of motion.

### 3.2 Local form of the metric

By analysing the differential conditions (3.4)–(3.7) on the forms, after some work, one can derive the necessary and sufficient conditions on the local form of the metric and flux. Here we will simply summarize the results referring to [42] for more details.

First one notes that as a vector  $e^{-\lambda} \cos \zeta K_2$  is Killing and that coordinates can be chosen so that

$$\frac{\partial}{\partial \psi} = \frac{1}{3m} e^{-\lambda} \cos \zeta K_2 \quad (3.9)$$

In addition the Lie derivatives  $\mathcal{L}_{\partial/\partial \psi} G = \mathcal{L}_{\partial/\partial \psi} \lambda = 0$  vanish so in fact acting with  $\mathcal{L}_{\partial/\partial \psi}$  preserves the full solution. This reflects the fact that the dual field theory has a  $U(1)_R$  symmetry.

Second, one can introduce a coordinate  $y$  for  $K_1$  given by

$$2my = e^{3\lambda} \sin \zeta \quad (3.10)$$

so that

$$K^1 = e^{-2\lambda} \sec \zeta \, dy. \quad (3.11)$$

While we could eliminate either  $\lambda$  or  $\zeta$  from the following formulae, for the moment it will be more convenient to keep both.

The metric then takes the form

$$g_X = e^{-4\lambda} (\hat{g} + \sec^2 \zeta \, dy \otimes dy) + \frac{1}{9m^2} e^{2\lambda} \cos^2 \zeta (d\psi + \rho) \otimes (d\psi + \rho) \quad (3.12)$$

where  $i_{\partial_y} \rho = i_{\partial_\psi} \rho = 0$ . We have, with  $\hat{J} = e^{4\lambda} J$ ,

$$(a) \quad \partial/\partial\psi \text{ is a Killing vector,} \quad (3.13)$$

$$(b) \quad \hat{g} \text{ is a family of Kähler metrics on } M_4 \text{ parameterized by } y, \quad (3.14)$$

$$(c) \quad \text{the corresponding complex structure } \hat{J}_i{}^j \text{ is independent of } y \text{ and } \psi, \quad (3.15)$$

and

$$(d) \quad 2my = e^{3\lambda} \sin \zeta, \quad (3.16)$$

$$(e) \quad \rho = \hat{P} + \hat{J} \cdot d_4 \log \cos \zeta, \quad (3.17)$$

where  $\hat{P}$  is the canonical connection defined by the Kähler metric, satisfying  $\hat{\mathfrak{R}} = d\hat{P}$  where  $\hat{\mathfrak{R}}$  is the Ricci form. Finally we have the conditions

$$(f) \quad \partial_y \hat{J} = -\frac{2}{3} y d_4 \rho, \quad (3.18)$$

$$(g) \quad \partial_y \log \sqrt{\det \hat{g}} = -3y^{-1} \tan^2 \zeta - 2\partial_y \log \cos \zeta. \quad (3.19)$$

Writing  $d = dy \wedge \partial/\partial y + d\psi \wedge \partial/\partial\psi + d_4$  and  $\hat{*}_4$  for the Hodge duality operator defined by  $\hat{g}$ , the four-form flux  $G$  is given by

$$\begin{aligned} G = & -(\partial_y e^{-6\lambda}) \widehat{\text{vol}}_4 - e^{-10\lambda} \sec \zeta (\hat{*}_4 d_4 e^{6\lambda}) \\ & \wedge K^1 - \frac{1}{3m} e^{-\lambda} \cos^3 \zeta (\hat{*}_4 \partial_y \rho) \wedge K^2 \\ & + e^\lambda \left[ \frac{1}{3m} \cos^2 \zeta \hat{*}_4 d_4 \rho - 4m e^{-6\lambda} \hat{J} \right] \wedge K^1 \wedge K^2 \end{aligned} \quad (3.20)$$

and is independent of  $\psi$  – that is,  $\mathcal{L}_{\partial/\partial\psi} G = 0$ . As discussed previously the equations of motion for  $G$  and the Bianchi identity are implied by expressions (3.13)–(3.19).

To summarize, we have given the local form of the generic  $\mathcal{N} = 1$  AdS<sub>5</sub> compactification in  $d = 11$  supergravity. Any  $d = 11$  AdS-CFT supergravity dual of a  $d = 4$  superconformal field theory will have this form.

### 3.3 Complex $X$ ansatz

In this section we consider how the conditions on the metric specialise for solutions where the six-dimensional space  $X$  is a complex manifold. Crucially, the supersymmetry conditions simplify considerably and we are able to find many solutions in closed form. Globally, the new regular compact solutions that we construct are all holomorphic  $\mathbb{C}P^1$  bundles over a smooth four-dimensional Kähler base  $M_4$ . Using a recent mathematical result on Kähler manifolds [2], we are able to classify completely this class of solutions (assuming that the Goldberg conjecture is true). In particular, at fixed  $y$  the base is either (i) a Kähler–Einstein (KE) space or (ii) a non-Einstein space which is the product of two constant curvature Riemann surfaces.

More precisely we specialize to the case where

$g_X$  is a Hermitian metric on a complex manifold  $X$ ,

where we define the complex structure, compatible with  $g_X$  and the local  $U(2)$ -structure, given by the holomorphic three-form  $\Omega_{(3)} = \Omega \wedge (K^1 + iK^2)$ . Requiring this complex structure to be integrable, that is  $d\Omega_{(3)} = A \wedge \Omega_{(3)}$  for some  $A$ , implies that

$$d_4\zeta = 0, \quad d_4\lambda = 0, \quad \partial_y\rho = 0. \quad (3.21)$$

In addition one finds that the connection  $\rho$  is simply the canonical connection defined by the Kähler metric  $\hat{g}$ , that is

$$\rho = \hat{P} \quad (3.22)$$

together with the useful condition that

at fixed  $y$ , the Ricci tensor on  $\text{Ric}_{\hat{g}}$  has two pairs of constant eigenvalues.

We would like to find global regular solutions for the complex manifold  $X$ . Our construction is as follows. We require that  $\psi$  and  $y$  describe a holomorphic  $\mathbb{C}P^1$  bundle over a smooth Kähler base  $M_4$

$$\begin{array}{ccc} \mathbb{C}P^1_{y,\psi} & \longrightarrow & X \\ & & \downarrow \\ & & M_4 \end{array} \quad (3.23)$$

For the  $(y, \psi)$  coordinates to describe a smooth  $\mathbb{C}P^1$  we take the Killing vector  $\partial/\partial\psi$  to have compact orbits so that  $\psi$  defines an azimuthal angle and  $y$  is taken to lie in the range  $[y_1, y_2]$  with  $\cos\zeta(y_i) = 0$ . Thus  $y_i$  are the two poles where the  $U(1)$  fibre shrinks to zero size. It turns out that the metric  $g_X$  gives a smooth  $S^2$  only if we choose the period of  $\psi$  to be  $2\pi$ . Given the connection (3.22), we see that, as a complex manifold,

$$X = \mathbb{P}(\mathcal{O} \oplus \mathcal{L}), \quad (3.24)$$

where  $\mathcal{L}$  is the canonical bundle and  $\mathcal{O}$  the trivial bundle on the base  $M_4$ .

Let us now consider the Kähler base. A recent result on Kähler manifolds (Theorem 2 of [2]) states that, if the Goldberg conjecture<sup>2</sup> is true, then a compact Kähler four-manifold whose Ricci tensor has two distinct pairs of constant eigenvalues is locally the product of two Riemann surfaces of (distinct) constant curvature. If the eigenvalues are the same the manifold is by definition Kähler–Einstein. The compactness in the theorem is essential, since there exist non-compact counterexamples. However, for AdS/CFT purposes, we are most interested in the compact case (for example, the central charge of the dual CFT is inversely proportional to the volume).

From now on we will consider only these two cases. One then finds that the conditions (3.18) and (3.19) can be partially integrated. In summary we have two cases:

$$\begin{aligned} \text{case 1: } \hat{g} &= \frac{1}{3}(b - ky^2)\tilde{g}_k, \\ \text{case 2: } \hat{g} &= \frac{1}{3}(a_1 - k_1y^2)\tilde{g}_{k_1} + \frac{1}{3}(a_2 - k_2y^2)\tilde{g}_{k_2} \end{aligned} \quad (3.25)$$

where  $k, k_i \in \{0, \pm 1\}$ , and the (two- or four-dimensional) Kähler–Einstein metrics  $\tilde{g}_k$  satisfy

$$\text{Ric}_{\tilde{g}_k} = k\tilde{g}_k \quad (3.26)$$

and are independent of  $y$ . The remaining equation (3.19), implies

$$\begin{aligned} \text{case 1: } m^2(1 + 6y\partial_y\lambda) &= \frac{k}{b - ky^2} (e^{6\lambda} - 4m^2y^2), \\ \text{case 2: } m^2(1 + 6y\partial_y\lambda) &= \frac{k_2a_1 + k_1a_2 - 2k_1k_2y^2}{2(a_1 - k_1y^2)(a_2 - k_2y^2)} (e^{6\lambda} - 4m^2y^2). \end{aligned} \quad (3.27)$$

### 3.4 New compact solutions

**3.4.1 Case 1. KE base.** We start by considering the case where the base is Kähler–Einstein (KE). The remaining supersymmetry condition (3.27) can be integrated explicitly. One finds

$$\begin{aligned} e^{6\lambda} &= \frac{2m^2(b - ky^2)^2}{2kb + cy + 2k^2y^2}, \\ \cos^2 \zeta &= \frac{b^2 - 6kby^2 - 2cy^3 - 3k^2y^4}{(b - ky^2)^2} \end{aligned} \quad (3.28)$$

where  $c$  is an integration constant. Without loss of generality by an appropriate rescaling of  $y$  we can set  $b = 1$  and  $c \geq 0$ .

Assuming  $X$  has the topology given by (3.24), we find this leads to a smooth metric at the  $y = y_i$  poles of the  $\mathbb{C}P^1$  fibres provided we take  $\psi$  to have period  $2\pi$ . One then finds our first result:

---

<sup>2</sup>The Goldberg conjecture says that any compact Einstein almost Kähler manifold is Kähler–Einstein, i.e. the complex structure is integrable. This has been proven for non-negative curvature [85].

*For  $0 \leq c < 4$  we have a one-parameter family of completely regular, compact, complex metrics  $g_X$  with the topology of a  $\mathbb{C}P^1$  fibration over a positive curvature KE base.*

For negative ( $k = -1$ ) and zero ( $k = 0$ ) curvature KE metrics  $\hat{g}$  there are no regular solutions. Since four-dimensional compact Kähler-Einstein spaces with positive curvature have been classified [88, 89], we have a classification for the above solutions. In particular, the base space is either  $S^2 \times S^2$  or  $\mathbb{C}P^2$ , or  $\mathbb{C}P^2 \#_n \mathbb{C}P^2$  with  $n = 3, \dots, 8$ . For the first two examples, the KE metrics are of course explicitly known and this gives explicit solutions of M-theory. The remaining metrics, although proven to exist, are not explicitly known, and so the same applies to the corresponding M-theory solutions.

**3.4.2 Case 2. Product base.** Next consider the case where the base is a product of constant curvature Riemann surfaces. Again the remaining supersymmetry condition (3.27) can be integrated explicitly giving

$$\begin{aligned} e^{6\lambda} &= \frac{2m^2(a_1 - k_1 y^2)(a_2 - k_2 y^2)}{(k_2 a_1 + k_1 a_2) + cy + 2k_1 k_2 y^2}, \\ \cos^2 \zeta &= \frac{a_1 a_2 - 3(k_2 a_1 + k_1 a_2)y^2 - 2cy^3 - 3k_1 k_2 y^4}{(a_1 - k_1 y^2)(a_2 - k_2 y^2)} \end{aligned} \quad (3.29)$$

where  $c$  is an integration constant giving a three-parameter family of solutions. Note that on setting  $a_1 = a_2 = b$ ,  $k_1 = k_2 = k$  these reduce to the KE solutions considered above. Again we have a smooth metric at the  $y = y_i$  poles of the  $\mathbb{C}P^2$  fibres provided we take  $\psi$  to have period  $2\pi$ . The full metric  $g_X$  is regular if the base is  $S^2 \times T^2$ ,  $S^2 \times S^2$  or  $S^2 \times H^2$ . However the final case is not compact.

Summarizing the compact cases, for  $S^2 \times T^2$  without loss of generality we can take  $k_2 = 0$ ,  $a_2 = 3$  and, by scaling  $y$ , we can set  $c = 1$  or  $c = 0$ . We find:

*For  $0 < a < 1$  and  $c \neq 0$  we have a one-parameter family of completely regular, compact, complex metrics  $g_X$  where  $X$  is a topologically trivial  $\mathbb{C}P^1$  bundle over  $S^2 \times T^2$ . A single additional solution of this type is obtained when  $c = 0$  and  $a \neq 0$ .*

For the  $S^2 \times S^2$  topology again, generically, one parameter can be scaled away and we find:

*For various ranges of  $(a_1, a_2, c)$  there are completely regular, compact, complex metrics  $g_X$  where  $X$  is topologically a  $\mathbb{C}P^1$  bundle over  $S^2 \times S^2$ .*

In particular there are solutions when  $a_1$  is not equal to  $a_2$  and hence this gives a broader class of solutions than in the Kähler-Einstein case considered above. The existence of regular solutions is rather easy to see if one sets  $c = 0$ . Note that we can also recover the well-known Maldacena–Nuñez solution [74] when the base has topology  $S^2 \times H^2$ , though the topology is slightly different from the ansatz here. More details are given in [42].

The  $S^2 \times T^2$  solutions are of particular interest since they lead to new type IIA and type IIB supergravity solutions. Type IIA supergravity arises from  $d = 11$  supergravity reduced on a circle. Since these solutions have two Killing directions on the  $T^2$  base we can trivially reduce on one circle in  $T^2$  to give a IIA solution. Given the second Killing vector we can then use T-duality to generate a IIB solution. (T-duality is a specific map between IIA and IIB supergravity backgrounds which exists when each background has a Killing vector which also preserves the flux and dilaton, and also the Killing spinors if the map is to preserve supersymmetry at the level of the supergravity solution.) The resulting IIB background has the form  $\text{AdS}_5 \times Z$  with non-trivial  $F^{(5)}$  flux. As we will see, this implies that  $Z$  is a Sasaki–Einstein manifold. The geometry of these manifolds will be the subject of the following section.

## 4 A new infinite class of Sasaki–Einstein solutions

By analogy with the previous section let us now turn to the case of  $\text{AdS}_5 \times X$  solutions in IIB supergravity with non-trivial  $F^{(5)}$ . It is a well-known result that  $X$  must then be Sasaki–Einstein [68]. As noted above, the  $d = 11$  solutions on  $S^2 \times T^2$  potentially give new  $n = 5$  Sasaki–Einstein solutions. In this section we discuss the structure of these solutions. In fact we will show the general result:

*For every positive curvature  $2n$ -dimensional Kähler–Einstein manifold  $B_{2n}$ , there is a countably infinite class of associated compact, simply-connected, spin, Sasaki–Einstein manifolds  $X_{2n+3}$  in dimension  $2n + 3$ .*

### 4.1 Sasaki–Einstein spaces

Let us start by showing directly that for IIB backgrounds of the form  $\text{AdS}_5 \times X$  with  $F^{(5)}$  flux  $X$  must be Sasaki–Einstein. As before we will consider the reduction from  $\mathbb{R}^{3,1} \times X'$  to  $\text{AdS}_5 \times X$  of the backgrounds given in eqns. (2.33). Let  $(J', \Omega')$  denote the  $\text{SU}(3)$  structure on  $X'$ . Picking out the radial one-form  $R \equiv e^\lambda dr$  globally defines a second one-form  $K = J' \cdot R$ . One then has the real and complex two-forms given by

$$\begin{aligned} J &= J' - K \wedge R \\ \Omega &= i_{K+iR}\Omega' . \end{aligned} \tag{4.1}$$

Note that globally  $\Omega$  is not strictly a two-form on  $X$  but is a section of  $\Lambda^2 T^*X$  twisted by the complex line bundle defined by  $K + iR$ . For this reason  $(K, J, \Omega)$  define only a  $\text{U}(2)$  (or almost metric contact) structure on  $X$  rather than  $\text{SU}(2)$ .



Reducing the condition on  $(J', \Omega')$  one finds that  $\lambda$  is constant and we set it to zero without loss of generality. We then have that  $K$  is unit norm and that

$$\begin{aligned} dK &= 2mJ \\ d\Omega &= i3mK \wedge \Omega \end{aligned} \tag{4.2}$$

with the five-form flux given by  $F^{(5)} = 4m(\text{vol}_{\text{AdS}_5} + \text{vol}_{X_5})$ . Clearly  $\mathcal{L}_K J = 0$  and  $\mathcal{L}_K \Omega = i3m\Omega$  so that  $K$  is a Killing vector. The second condition in (4.2) implies that we have an integrable contact structure. The first condition implies that the metric is actually Sasaki–Einstein. (For more details see for example [46] and [16].)

The Killing condition means that locally we have

$$K = d\psi' + \sigma \tag{4.3}$$

where  $d\sigma = 2mJ$  and that we can write the metric in the form

$$g_X = \hat{g} + K \otimes K \tag{4.4}$$

where  $\hat{g}$  is a positive curvature Kähler–Einstein metric. Note that, by definition, the metric cone over  $g_X$  is Calabi–Yau. All these results generalize without modification to  $(2k+1)$ -dimensional Sasaki–Einstein manifolds  $X$ .

Finally note that one can group Sasaki–Einstein manifolds by the nature of the orbits of the Killing vector  $K$ . If the orbits close, then we have a  $U(1)$  action. Since  $K$  is nowhere vanishing, it follows that the isotropy groups of this action are all finite. Thus the space of leaves of the foliation will be a positive curvature Kähler–Einstein orbifold of complex dimension  $k$ . Such Sasaki–Einstein manifolds are called quasi-regular. If the  $U(1)$  action is free, the space of leaves is actually a Kähler–Einstein manifold and the Sasaki–Einstein manifold is then said to be regular. Moreover, the converse is true: there is a Sasaki–Einstein structure on the total space of a certain  $U(1)$  bundle over any given Kähler–Einstein manifold of positive curvature [70]. A similar result is true in the quasi-regular case [15]. If the orbits of  $K$  do not close, the Sasaki–Einstein manifold is said to be irregular.

Although there are many results in the literature on Sasaki–Einstein manifolds explicit metrics are rather rare. Homogeneous regular Sasaki–Einstein manifolds are classified: they are all  $U(1)$  bundles over generalized flag manifolds [14]. This result follows from the classification of homogeneous Kähler–Einstein manifolds. Inhomogeneous Kähler–Einstein manifolds are known to exist and so one may then construct the associated regular Sasaki–Einstein manifolds. However, until recently, there have been no known explicit inhomogeneous simply-connected<sup>3</sup> manifolds in the quasi-regular class. Moreover, no irregular examples were known at all.

The family of solutions we construct in the following thus gives not only the first explicit examples of inhomogeneous quasi-regular Sasaki–Einstein manifolds, but also the first examples of irregular geometries.

---

<sup>3</sup>One can obtain quasi-regular geometries rather trivially by taking a quotient of a regular Sasaki–Einstein manifold by an appropriate finite freely-acting group. Our definition of Sasaki–Einstein will always mean simply-connected.

## 4.2 The local metric

Let  $B$  be a (complete)  $2n$ -dimensional positive curvature Kähler–Einstein manifold, with metric  $g_B$  and Kähler form  $J_B$  such that  $\text{Ric}_{g_B} = \lambda g_B$  with  $\lambda > 0$ . It is thus necessarily compact [80] and simply-connected [70]. We construct the local Sasaki–Einstein metric (4.4) in two steps. First, following [12] and [81], consider the local  $2n + 2$ -dimensional metric

$$\hat{g} = \rho^2 g_B + U^{-1} d\rho \otimes d\rho + \rho^2 U (d\tau - A) \otimes (d\tau - A) \quad (4.5)$$

where

$$U(\rho) = \frac{\lambda}{2n+2} - \frac{\Lambda}{2n+4} \rho^2 + \frac{\Lambda}{2(n+1)(n+2)} \left( \frac{\lambda}{\Lambda} \right)^{n+2} \frac{\kappa}{\rho^{2n+2}}, \quad (4.6)$$

$\kappa$  is a constant,  $\Lambda > 0$  and

$$dA = 2J_B, \quad (4.7)$$

or, in other words, we can take  $A = 2P_B/\lambda$  where  $P_B$  is the canonical connection defined by  $J_B$ . By construction  $\hat{g}$  is a positive curvature Kähler–Einstein metric with

$$\hat{J} = \rho^2 J_B + \rho (d\tau - A) \wedge d\rho \quad (4.8)$$

and  $\text{Ric}_{\hat{g}} = \Lambda \hat{g}$ . (Clearly  $\hat{J}$  is closed. If we let  $\hat{\Omega}$  be the corresponding  $(n+1, 0)$  form, then we calculate  $d\hat{\Omega} = i\hat{P} \wedge \hat{\Omega}$ , leading to a Ricci-form given by  $\hat{\mathcal{R}} \equiv d\hat{P} = \Lambda \hat{J}$ .)

In [81] it was shown that the local expression (4.5) describes a complete metric on a manifold if and only if  $\kappa = 0$ ,  $B$  is  $\mathbb{C}P^n$  and the total space is  $\mathbb{C}P^{n+1}$  the latter each with the canonical metric. Here we consider adding another dimension to the metric above – specifically, the local Sasaki–Einstein direction. We define the  $(2n+3)$ -dimensional local metric, as in (4.4)

$$g_X = \hat{g} + (d\psi' + \sigma) \otimes (d\psi' + \sigma) \quad (4.9)$$

where  $d\sigma = 2\hat{J}$ . As is well-known (see for example [46] for a recent review), such a metric is locally Sasaki–Einstein. The curvature is  $2n+2$ , provided  $\Lambda = 2(n+2)$ . An appropriate choice for the connection one-form  $\sigma$  is

$$\sigma = \frac{\lambda}{\Lambda} A + \left( \frac{\lambda}{\Lambda} - \rho^2 \right) (d\tau - A). \quad (4.10)$$

By a rescaling we can, and often will, set  $\lambda = 2$ .

## 4.3 Global analysis

We next show that the metrics (4.9) give an infinite family of complete, compact Sasaki–Einstein metrics on a  $2n+3$ -dimensional space  $X$ . Topologically  $X$  will be given by  $S^1$  bundles over  $\mathbb{P}(\mathcal{O} \oplus \mathcal{L}_B)$  where  $\mathcal{O}$  is the trivial bundle and  $\mathcal{L}_B$  the canonical bundle on  $B$ . However, it should be noted that the complex structure of

the Calabi–Yau cone is *not* compatible with that on  $\mathbb{P}(\mathcal{O} \oplus \mathcal{L}_B)$  – we use the latter notation only as a convenient way to represent the topology.

The first step is to make a very useful change of coordinates which casts the local metric (4.9) into a different  $(2n + 2) + 1$  decomposition. Define the new coordinates

$$\alpha = -\tau - \frac{\Lambda}{\lambda} \psi' \quad (4.11)$$

and  $(\Lambda/\lambda)\psi' = \psi$ . We then have

$$g_X = \rho^2 g_B + U^{-1} d\rho \otimes d\rho + q(d\psi + A) \otimes (d\psi + A) + w(d\alpha + C) \otimes (d\alpha + C) \quad (4.12)$$

where

$$\begin{aligned} q(\rho) &= \frac{\lambda^2}{\Lambda^2} \frac{\rho^2 U(\rho)}{w(\rho)}, \\ w(\rho) &= \rho^2 U(\rho) + (\rho^2 - \lambda/\Lambda)^2, \\ C &= f(r)(d\psi + A). \end{aligned} \quad (4.13)$$

and

$$f(r) \equiv \frac{\rho^2(U(\rho) + \rho^2 - \lambda/\Lambda)}{w(\rho)}. \quad (4.14)$$

The metric is Riemannian only if  $U \geq 0$  and hence  $w \geq 0$  and  $q \geq 0$ . This implies that we choose the range of  $\rho$  to be

$$\rho_1 \leq \rho \leq \rho_2 \quad (4.15)$$

where  $\rho_i$  are two appropriate roots of the equation  $U(\rho) = 0$ . As we want to exclude  $\rho = 0$ , since the metric is generically singular there, we thus take  $\rho_i$  to be both positive (without loss of generality). Considering the roots of  $U(\rho)$  we see that we need only consider the range

$$-1 < \kappa \leq 0 \quad (4.16)$$

so that

$$0 \leq \rho_1 < \sqrt{\frac{\lambda}{\Lambda}} < \rho_2 \leq \sqrt{\frac{\lambda(n+2)}{\Lambda(n+1)}}. \quad (4.17)$$

The limiting value  $\kappa = 0$  in (4.5) gives a smooth compact Kähler–Einstein manifold if and only if  $B = \mathbb{C}P^n$ , in which case  $X$  is  $S^n$  (or a discrete quotient thereof). As a consequence, we can focus on the case where the range of  $\kappa$  is  $-1 < \kappa < 0$ .

Topologically we want  $X$  to be a  $S^1$  fibration over  $Y = \mathbb{P}(\mathcal{O} \oplus \mathcal{L}_B)$ .

$$\begin{array}{ccc} S^1_\alpha & \longrightarrow & X \\ \downarrow & & \downarrow \\ & & Y \end{array} \quad \begin{array}{ccc} S^2_{\rho, \psi} & \longrightarrow & Y \\ \downarrow & & \downarrow \\ & & B \end{array} \quad (4.18)$$

As indicated, we construct the  $S^1$  fibre from the  $\alpha$  coordinate and the  $S^2$  fibre of the base bundle  $Y$  from the  $(\rho, \psi)$  coordinates, where  $\psi$  is the azimuthal angle and  $\rho = \rho_i$  are the north and south poles.

To make this identification we first need to check that the metric is smooth at the poles  $\rho = \rho_i$ . It is easy to show that this is true provided  $\psi$  has period  $2\pi$  (with  $\lambda = 2$  and  $\Lambda = 2(n+2)$ ).

Next consider the  $X \rightarrow Y$  circle fibration. Suppose  $\alpha$  has period  $2\pi\ell$ . The question is then, can we choose  $\ell$  and the parameter  $\kappa$  such that  $g_X$  is a regular globally defined metric on  $X$ . The only point to check is that the term  $C$  in the expression for the metric (4.12) is in fact a connection on a  $U(1)$  bundle. The necessary and sufficient condition is that the periods of the corresponding curvature are integral, that is

$$\frac{1}{2\pi}F = \frac{1}{2\pi\ell}dC \in H_{\text{de Rham}}^2(Y, \mathbb{Z}). \quad (4.19)$$

(Note that since  $B$  is simply connected, so is  $Y$  and hence also  $H^2(Y, \mathbb{Z})$  is torsion free. Thus, the periods of  $F$  in fact completely determine the  $U(1)$  bundle.)

To check the condition (4.19) we need a basis for the torsion-free part of  $H_2(Y, \mathbb{Z})$ . First note that since  $Y$  is a projectivised bundle over  $B$ , we can use the results of Section 20 of [13] to write down the cohomology ring of  $Y$  in terms of  $B$ . In particular, we have  $H^2(Y, \mathbb{Z}) \cong \mathbb{Z} \oplus H^2(B, \mathbb{Z})$ , where the first factor is generated by the cohomology class of the  $S^2$  fibre. Let  $\{\Sigma_i\}$  be a set of two-cycles in  $B$  such that the homology classes  $[\Sigma_i]$  generate the torsion-free part of  $H_2(B, \mathbb{Z})$ . Next define a submanifold  $\Sigma \cong S^2$  of  $Y$  corresponding to the fibre of  $Y$  at some fixed point on the base  $B$ . Finally we also have the global section  $\sigma^N: B \rightarrow Y$  corresponding to the “north pole” ( $\rho = \rho_1$ ) of the  $S^2$  fibres. Together we can then construct the set  $\{\Sigma, \sigma^N \Sigma_i\}$  which forms a representative basis generating the free part of  $H_2(Y, \mathbb{Z})$ .

Calculating the periods of  $F$  we find

$$\begin{aligned} \int_{\Sigma} \frac{F}{2\pi} &= \frac{f(\rho_1) - f(\rho_2)}{\ell}, \\ \int_{\sigma^N \Sigma_i} \frac{F}{2\pi} &= \frac{f(\rho_2)c_{(i)}}{\ell}, \end{aligned} \quad (4.20)$$

where

$$c_{(i)} = \int_{\Sigma_i} \frac{dA}{2\pi} = \langle c_1(\mathcal{L}_B), [\Sigma_i] \rangle \in \mathbb{Z} \quad (4.21)$$

are the periods of the canonical bundle  $\mathcal{L}_B$ . Thus we have integral periods if and only if  $f(\rho_1)/f(\rho_2) = p/q \in \mathbb{Q}$ , with  $p, q \in \mathbb{Z}$  and  $\ell = f(\rho_2)/q = f(\rho_1)/p$ . The periods of  $\frac{1}{2\pi}F$  are then  $\{p - q, qc_{(i)}\}$ . Rescaling  $\ell$  by  $h = \text{hcf}\{p - q, qc_{(i)}\}$  gives a special class of solutions where the integral periods  $\{h^{-1}(p - q), h^{-1}qc_{(i)}\}$  have no common factor. This is the class we will concentrate on from now on since in that case  $X$  is simply-connected (see [43], [44]).

Notice that, using the expression (4.13), we have

$$R(\kappa) \equiv \frac{f(\rho_1)}{f(\rho_2)} = \frac{\rho_1^2(\rho_2^2 - \lambda/\Lambda)}{\rho_2^2(\rho_1^2 - \lambda/\Lambda)}. \quad (4.22)$$

This is a continuous function of  $\kappa$  in the interval  $(-1, 0]$ . Moreover, it is easy to see that  $R(0) = 0$  and  $R(-1) = -1$ . Hence there are clearly a countably infinite number of values of  $\kappa$  for which  $R(\kappa)$  is rational and equal to  $p/q$ , with  $|p/q| < 1$ , and these all give complete Riemannian metrics  $g_X$ .

Locally, by construction,  $g_X$  was a Sasaki–Einstein metric. In the  $(\alpha, \psi)$  coordinates the unit-norm Killing vector  $K$  is given by

$$K = \frac{\Lambda}{\lambda} \left( \frac{\partial}{\partial \psi} - \frac{\partial}{\partial \alpha} \right). \quad (4.23)$$

Given the topology of  $X$  it is easy to see that this is globally defined. Hence, so is the corresponding one-form and also  $J = \frac{1}{2}dK$ . Thus the Sasaki–Einstein structure is globally defined.

Recall that our original problem was to find  $X$  which admitted Killing spinors. For this we need  $X$  to be spin. However, by construction this is the case irrespective of whether  $B$  is spin or not (see [44]). Since  $X$  is also simply-connected we can then invoke theorem 3 of [34] to see that this implies we have global Killing spinors.

Finally we notice that the orbits of the Killing vector  $K$  close if and only if  $f(\rho_2) \in \mathbb{Q}$ , in which case the Sasaki–Einstein manifold is quasi-regular. For generic  $p$  and  $q$  the space will be irregular. Determining when  $f(\rho_2) \in \mathbb{Q}$  seems to be a non-trivial number-theoretic problem. (Though for the case  $n = 1$  studied in [43] one has to solve a quadratic diophantine, which can be done using standard methods.) Thus for the countably infinite number of values of  $\kappa$  found here, the Kähler–Einstein “base” is at best an orbifold, and in the irregular case there is in fact no well-defined base at all.

## 4.4 Five-dimensional solutions

Of particular interest in string theory are five-dimensional Sasaki–Einstein solutions. For our class these are the backgrounds which are dual to the  $S^2 \times T^2$  solutions in  $d = 11$  supergravity discussed in the previous section.

To make the correspondence explicit we take  $n = 1$ ,  $\lambda = 2$  and  $\Lambda = 6$  and introduce the new coordinate  $\rho^2 = (\lambda/\Lambda)(1 - cy)$ . The metric (4.12) then takes the form

$$g_X = \frac{1}{6}(1 - cy)\tilde{g} + w^{-1}q^{-1}dy \otimes dy + \frac{q}{9}(d\psi + \tilde{P}) \otimes (d\psi + \tilde{P}) + w(d\alpha + C) \otimes (d\alpha + C) \quad (4.24)$$

where

$$\begin{aligned} q(y) &= \frac{a - 3y^2 + 2cy^3}{a - y^2}, \\ w(y) &= \frac{2(a - y^2)}{1 - cy}, \\ C &= \frac{ac - 2y + cy^2}{6(a - y^2)}(d\psi + \tilde{P}). \end{aligned} \quad (4.25)$$

Here  $\tilde{g}$  is the canonical metric on  $S^2$  with  $\text{Ric}_{\tilde{g}} = \tilde{g}$ , and  $\tilde{P}$  is the corresponding canonical connection. This is the form of the metric one obtains by making an explicit duality from the  $d = 11$  solutions given in the previous section.

Topologically these spaces are all  $S^2 \times S^3$  [43]. One finds both quasi-regular and irregular solutions depending on whether or not  $f(\rho_2) \in \mathbb{Q}$ . In the dual field theory this corresponds to rational or irrational R-charges and also central charge. Interestingly the irrational charges are at most quadratic algebraic, that is can be written in terms of square-roots of rational numbers. This matches a field theory argument due to Intriligator and Wecht [64].

## 4.5 A simple generalisation

We end by noting that there is a simple generalisation of the above construction to the case when the base manifold  $B$  is a product of Kähler–Einstein manifolds.

More specifically, let  $g_i$  with  $i = 1, \dots, p$  be a set of  $2n_i$ -dimensional positive curvature Kähler–Einstein metrics with Kähler forms  $J_i$  and  $\text{Ric}_{g_i} = \lambda_i g_i$ . There is then a straightforward generalisation of the construction of [12] and [81] allowing one to build a  $(2n + 2)$ -dimensional Kähler–Einstein metric  $\hat{g}$  where  $n = \sum_i n_i$ . In analogy with (4.5) we write

$$\hat{g} = \sum_i \left( r^2 + \frac{\lambda_i}{\Lambda} \right) g_i + V(r)^{-1} dr \otimes dr + r^2 V(r) (d\tau - A) \otimes (d\tau - A) \quad (4.26)$$

and define the corresponding fundamental form

$$\hat{J} = \sum_i \left( r^2 + \frac{\lambda_i}{\Lambda} \right) J_i + r (d\tau - A) \wedge dr. \quad (4.27)$$

The metric is Kähler–Einstein with  $\text{Ric}_{\hat{g}} = \Lambda \hat{g}$ , provided that

$$\begin{aligned} dA &= 2 \sum_i J_i, \\ r^2 V(r) &= -\frac{1}{2} \Lambda f(r^2; \lambda_i/\Lambda, \mu), \end{aligned} \quad (4.28)$$

where

$$f(x; d_i, \mu) = \frac{\mu + \int_0^x dx' x' \prod_i (x' + d_i)^{n_i}}{\prod_i (x + d_i)^{n_i}} \quad (4.29)$$

and  $\mu$  is an integration constant. Note that we have chosen a slightly different convention for the coordinate  $r$  as compared to the case of  $p = 1$  given in (4.6). They differ by  $r^2 = \rho^2 - \lambda/\Lambda$ .

Using the construction of Section 4.2 above we can then obtain  $(2n+3)$ -dimensional metrics which are locally Sasaki–Einstein. For certain values of the constants  $\lambda_i$  and  $\mu$  one expects that these will give metrics on complete compact Sasaki–Einstein manifolds: the analysis will be a direct generalisation of that in [44] but we leave the details for future work.

The case of most interest for M-theory is when the dimension of the resulting Sasaki–Einstein manifold is seven, as they can be used to obtain new  $\text{AdS}_4 \times X$  solutions of  $d = 11$  supergravity. It was shown in [44] that the construction using (4.5) in case where the base is a direct product of two equal radius two-spheres generalises the well-known homogeneous Sasaki–Einstein metric  $Q^{1,1,1}$  (see for example [29] for a review). A further generalisation to the case where the spheres have different radii can be obtained from the generalised construction. Setting  $p = 2$ ,  $n_1 = n_2 = 1$  (so that  $g_i$  are round metrics on two-spheres) in (4.26) we obtain

$$\begin{aligned} \hat{g} = & \frac{1}{\Lambda} [(1 + c_1 x) \lambda_1 g_1 + (1 + c_2 x) \lambda_2 g_2] \\ & + \frac{dx \otimes dx}{F(x)} + \frac{F(x)}{\Lambda^2} (d\beta - c_i A_i) \otimes (d\beta - c_i A_i), \end{aligned} \quad (4.30)$$

where  $c_i = \Lambda/\lambda_i$ ,  $dA_i = \lambda_i J_i$  and

$$F(x) = -\frac{\Lambda}{8} \frac{16c_1 c_2 \mu + 8x^2 + \frac{16}{3}(c_1 + c_2)x^3 + 4c_1 c_2 x^4}{(1 + c_1 x)(1 + c_2 x)}. \quad (4.31)$$

**Acknowledgements.** JFS is supported by NSF grants DMS-0244464, DMS-0074239 and DMS-9803347. DJW is support by a Royal Society University Research Fellowship.

## References

- [1] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, Large  $N$  field theories, string theory and gravity, *Phys. Rep.* 323 (2000), 183–386 [arXiv:hep-th/9905111].
- [2] V. Apostolov, T. Draghici, and A. Moroianu, A splitting theorem for Kähler manifolds whose Ricci tensors have constant eigenvalues, *math.DG/0007122*.
- [3] C. Bachas, A way to break supersymmetry, *arXiv:hep-th/9503030*.
- [4] K. Becker and M. Becker, M-theory on eight-manifolds, *Nuclear Phys. B* 477 (1996), 155–167 [arXiv:hep-th/9605053].

- [5] K. Becker, M. Becker, K. Dasgupta and P. S. Green, Compactifications of heterotic theory on non-Kähler complex manifolds. I, *J. High Energy Phys.* 04 (2003), 007 [arXiv:hep-th/0301161]; K. Becker, M. Becker, P. S. Green, K. Dasgupta and E. Sharpe, Compactifications of heterotic strings on non-Kähler complex manifolds. II, *Nuclear Phys. B* 678 (2004), 19 [arXiv:hep-th/0310058].
- [6] K. Becker, M. Becker, K. Dasgupta and S. Prokushkin, Properties of heterotic vacua from superpotentials, *Nuclear Phys. B* 666 (2003), 144–174 [arXiv:hep-th/0304001].
- [7] K. Behrndt and M. Cvetič, General  $\mathcal{N} = 1$  supersymmetric flux vacua of (massive) type IIA string theory, arXiv:hep-th/0403049.
- [8] K. Behrndt and M. Cvetič, General  $\mathcal{N} = 1$  supersymmetric fluxes in massive type IIA string theory, arXiv:hep-th/0407263.
- [9] K. Behrndt and C. Jeschek, Fluxes in M-theory on 7-manifolds: G-structures and superpotential, *Nuclear Phys. B* 694 (2004), 99–114 [arXiv:hep-th/0311119].
- [10] K. Behrndt and C. Jeschek, Fluxes in M-theory on 7-manifolds and G-structures, *J. High Energy Phys.* 04 (2003), 002 [arXiv:hep-th/0302047].
- [11] K. Behrndt and C. Jeschek, Fluxes in M-theory on 7-manifolds:  $G_2$ ,  $SU(3)$  and  $SU(2)$  structures, arXiv:hep-th/0406138.
- [12] L. Bérard-Bergery, Sur de nouvelles variétés riemanniennes d'Einstein, Institut Élie Cartan, Université Nancy I, 6 (1983), 1–60.
- [13] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, Springer-Verlag, New York 1982.
- [14] C. P. Boyer, K. Galicki, 3-Sasakian Manifolds, in: Surveys in differential geometry: essays on Einstein manifolds, *Surv. Differ. Geom.* 6 (1999), 123–184 [arXiv:hep-th/9810250].
- [15] C. P. Boyer and K. Galicki, On Sasakian-Einstein Geometry, *Internat. J. Math.* 11 (2000), 873–909 [arXiv:math.dg/9811098].
- [16] C. P. Boyer and K. Galicki, Sasakian Geometry, Hypersurface Singularities, and Einstein Metrics, arXiv:math.dg/0405256.
- [17] M. M. Caldarelli and D. Klemm, All supersymmetric solutions of  $\mathcal{N} = 2$ ,  $D = 4$  gauged supergravity, *J. High Energy Phys.* 09 (2003), 019 [arXiv:hep-th/0307022]; S. L. Cacciatori, M. M. Caldarelli, D. Klemm and D. S. Mansi, More on BPS solutions of  $\mathcal{N} = 2$ ,  $D = 4$  gauged supergravity, *J. High Energy Phys.* 07 (2004), 061 [arXiv:hep-th/0406238].
- [18] G. L. Cardoso, G. Curio, G. Dall'Agata and D. Lust, BPS action and superpotential for heterotic string compactifications with fluxes, *J. High Energy Phys.* 10 (2003), 004 [arXiv:hep-th/0306088].
- [19] G. L. Cardoso, G. Curio, G. Dall'Agata, D. Lust, P. Manousselis and G. Zoupanos, Non-Kähler string backgrounds and their five torsion classes, *Nuclear Phys. B* 652 (2003), 5–34 [arXiv:hep-th/0211118].
- [20] M. Cariglia and O. A. P. Mac Conamhna, Timelike Killing spinors in seven dimensions, arXiv:hep-th/0407127.
- [21] J. F. G. Cascales and A. M. Uranga, Branes on generalized calibrated submanifolds, arXiv:hep-th/0407132.



- [22] H. Cho, M. Emam, D. Kastor and J. H. Traschen, Calibrations and Fayyazuddin–Smith spacetimes, *Phys. Rev. D* 63 (2001), 064003 [arXiv:hep-th/0009062].
- [23] J. Dadok and R. Harvey, Calibrations and spinors, *Acta Math.* 170 (1993) 83–120; R. Harvey, *Spinors and calibrations*, Perspect. Math. 9, Academic Press Inc., Boston, MA, 1990.
- [24] G. Dall’Agata, On supersymmetric solutions of type IIB supergravity with general fluxes, *Nuclear Phys. B* 695 (2004), 243–266 [arXiv:hep-th/0403220].
- [25] G. Dall’Agata and N. Prezas,  $N = 1$  geometries for M-theory and type IIA strings with fluxes, *Phys. Rev. D* 69 (2004), 066004 [arXiv:hep-th/0311146].
- [26] K. Dasgupta, G. Rajesh and S. Sethi, M theory, orientifolds and G-flux, *J. High Energy Phys.* 08 (1999), 023 [arXiv:hep-th/9908088].
- [27] B. de Wit, D. J. Smit and N. D. Hari Dass, Residual supersymmetry of compactified  $D = 10$  supergravity, *Nuclear Phys. B* 283 (1987), 165.
- [28] M. J. Duff and J. T. Liu, Hidden spacetime symmetries and generalized holonomy in M-theory, *Nuclear Phys. B* 674 (2003), 217–230 [arXiv:hep-th/0303140].
- [29] M. J. Duff, B. E. W. Nilsson and C. N. Pope, Kaluza–Klein Supergravity, *Phys. Rep.* 130 (1986), 1–142.
- [30] S. Fianza, R. Minasian and A. Tomasiello, Mirror symmetric  $SU(3)$ -structure manifolds with NS fluxes, arXiv:hep-th/0311122; U. Lindström, R. Minasian, A. Tomasiello and M. Zabzine, Generalized complex manifolds and supersymmetry, arXiv:hep-th/0405085; M. Grana, R. Minasian, M. Petrini and A. Tomasiello, Supersymmetric backgrounds from generalized Calabi–Yau manifolds, *J. High Energy Phys.* 08 (2004), 046 [arXiv:hep-th/0406137].
- [31] J. Figueroa-O’Farrill and G. Papadopoulos, Maximally supersymmetric solutions of ten- and eleven-dimensional supergravities, *J. High Energy Phys.* 03 (2003), 048 [arXiv:hep-th/0211089].
- [32] A. R. Frey, Notes on  $SU(3)$  structures in type IIB supergravity, *J. High Energy Phys.* 06 (2004), 027 [arXiv:hep-th/0404107].
- [33] A. R. Frey and M. Grana, Type IIB solutions with interpolating supersymmetries, *Phys. Rev. D* 68 (2003), 106002 [arXiv:hep-th/0307142].
- [34] Th. Friedrich and I. Kath, Einstein manifolds of dimension five with small first eigenvalue of the Dirac operator, *J. Differential Geom.* 29 (1989), 263–279.
- [35] T. Friedrich and S. Ivanov, Parallel spinors and connections with skew-symmetric torsion in string theory, *Asian J. Math.* 6 (2002), 303–336 [arXiv:math.dg/0102142]; T. Friedrich and S. Ivanov, Killing spinor equations in dimension 7 and geometry of integrable  $G_2$ -manifolds, *J. Geom. Phys.* 48 (2003), 1–11 [arXiv:math.dg/0112201]; S. Ivanov, Connections with torsion, parallel spinors and geometry of  $Spin(7)$  manifolds, *Math. Res. Lett.* 11 (2004), 171–186 [arXiv:math.dg/0111216].
- [36] J. P. Gauntlett, Branes, calibrations and supergravity, arXiv:hep-th/0305074.
- [37] J. P. Gauntlett and S. Pakis, The geometry of  $D = 11$  Killing spinors, *J. High Energy Phys.* 04 (2003), 039 [arXiv:hep-th/0212008]; J. P. Gauntlett, J. B. Gutowski and S. Pakis, The geometry of  $D = 11$  null Killing spinors, *J. High Energy Phys.* 12 (2003), 049

- [arXiv:hep-th/0311112].
- [38] J. P. Gauntlett, D. Martelli and D. Waldram, Superstrings with intrinsic torsion, *Phys. Rev. D* 69 (2004), 086002 [arXiv:hep-th/0302158].
  - [39] J. P. Gauntlett, N. Kim, D. Martelli and D. Waldram, Fivebranes wrapped on SLAG three-cycles and related geometry, *J. High Energy Phys.* 11 (2001), 018 [arXiv:hep-th/0110034].
  - [40] J. P. Gauntlett, N. Kim, S. Pakis and D. Waldram, Membranes wrapped on holomorphic curves, *Phys. Rev. D* 65 (2002), 026003 [arXiv:hep-th/0105250].
  - [41] J. P. Gauntlett, D. Martelli, S. Pakis and D. Waldram, G-structures and wrapped NS5-branes, *Comm. Math. Phys.* 247 (2004), 421–445 [arXiv:hep-th/0205050].
  - [42] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, Supersymmetric  $\text{AdS}_5$  solutions of M-theory, *Classical Quantum Gravity* 21 (2004), 4335–4366 [arXiv:hep-th/0402153].
  - [43] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, Sasaki-Einstein metrics on  $S^2 \times S^3$ , arXiv:hep-th/0403002.
  - [44] J. P. Gauntlett, D. Martelli, J. F. Sparks and D. Waldram, A new infinite class of Sasaki-Einstein manifolds, arXiv:hep-th/0403038.
  - [45] J. P. Gauntlett, J. B. Gutowski, C. M. Hull, S. Pakis and H. S. Reall, All supersymmetric solutions of minimal supergravity in five dimensions, *Classical Quantum Gravity* 20 (2003), 4587–4634 [arXiv:hep-th/0209114]; J. P. Gauntlett and J. B. Gutowski, All supersymmetric solutions of minimal gauged supergravity in five dimensions, *Phys. Rev. D* 68 (2003), 105009 [arXiv:hep-th/0304064].
  - [46] G. W. Gibbons, S. A. Hartnoll and C. N. Pope, Bohm and Einstein–Sasaki metrics, black holes and cosmological event horizons, *Phys. Rev. D* 67 (2003), 084024 [arXiv:hep-th/0208031].
  - [47] J. Gillard, U. Gran and G. Papadopoulos, The spinorial geometry of supersymmetric backgrounds, arXiv:hep-th/0410155.
  - [48] E. Goldstein and S. Prokushkin, Geometric model for complex non-Kähler manifolds with  $\text{SU}(3)$  structure, arXiv:hep-th/0212307.
  - [49] C. N. Gowdigere, D. Nemeschansky and N. P. Warner, Supersymmetric solutions with fluxes from algebraic Killing spinors, *Adv. Theor. Math. Phys.* 7 (2004), 787–806 [arXiv:hep-th/0306097]; K. Pilch and N. P. Warner,  $\mathcal{N} = 1$  supersymmetric solutions of IIB supergravity from Killing spinors, arXiv:hep-th/0403005; D. Nemeschansky and N. P. Warner, A family of M-theory flows with four supersymmetries, arXiv:hep-th/0403006.
  - [50] M. Grana and J. Polchinski, Gauge/gravity duals with holomorphic dilaton, *Phys. Rev. D* 65 (2002), 126005 [arXiv:hep-th/0106014].
  - [51] S. Gukov, Solitons, superpotentials and calibrations, *Nuclear Phys. B* 574 (2000), 169–188 [arXiv:hep-th/9911011]; K. Behrndt and S. Gukov, Domain walls and superpotentials from M theory on Calabi–Yau three-folds, *Nuclear Phys. B* 580 (2000), 225–242 [arXiv:hep-th/0001082].
  - [52] S. Gukov, C. Vafa and E. Witten, CFT’s from Calabi–Yau four-folds, *Nuclear Phys. B* 584 (2000), 69–108; Erratum, *Nuclear Phys. B* 608 (2001), 477–478 [arXiv:hep-th/9906070].

- [53] S. Gurrieri and A. Micu, Type IIB theory on half-flat manifolds, *Classical Quantum Gravity* 20 (2003), 2181–2192 [arXiv:hep-th/0212278].
- [54] S. Gurrieri, A. Lukas and A. Micu, Heterotic on half-flat, arXiv:hep-th/0408121.
- [55] S. Gurrieri, J. Louis, A. Micu and D. Waldram, Mirror symmetry in generalized Calabi–Yau compactifications, *Nuclear Phys. B* 654 (2003), 61–113 [arXiv:hep-th/0211102].
- [56] J. Gutowski and G. Papadopoulos, AdS calibrations, *Phys. Lett. B* 462 (1999), 81–88 [arXiv:hep-th/9902034]; J. Gutowski, G. Papadopoulos and P. K. Townsend, Supersymmetry and generalized calibrations, *Phys. Rev. D* 60 (1999), 106006 [arXiv:hep-th/9905156].
- [57] J. B. Gutowski, D. Martelli and H. S. Reall, All supersymmetric solutions of minimal supergravity in six dimensions, *Classical Quantum Gravity* 20 (2003), 5049–5078 [arXiv:hep-th/0306235]; M. Cariglia and O. A. P. Mac Conamhna, The general form of supersymmetric solutions of  $\mathcal{N} = (1, 0)$  U(1) and SU(2) gauged supergravities in six dimensions, *Classical Quantum Gravity* 21 (2004), 3171–3195 [arXiv:hep-th/0402055].
- [58] E. J. Hackett-Jones, D. C. Page and D. J. Smith, Topological charges for branes in M-theory, *J. High Energy Phys.* 10 (2003), 005 [arXiv:hep-th/0306267]. E. J. Hackett-Jones and D. J. Smith, Type IIB Killing spinors and calibrations, arXiv:hep-th/0405098.
- [59] R. Harvey and H. B. Lawson, Calibrated geometries, *Acta Math.* 148 (1982), 47–157.
- [60] T. House and A. Lukas,  $G_2$  domain walls in M-theory, arXiv:hep-th/0409114.
- [61] C. M. Hull, Superstring compactifications with torsion and space-time supersymmetry, in: *Superunification and extra dimensions* (Torino, 1985), World Sci. Publishing, Singapore 1986, 347–375; Compactifications of the heterotic superstring, *Phys. Lett. B* 178 (1986), 357–364.
- [62] C. Hull, Holonomy and symmetry in M-theory, arXiv:hep-th/0305039; G. Papadopoulos and D. Tsimpis, The holonomy of the supercovariant connection and Killing spinors, *J. High Energy Phys.* 07 (2003), 018 [arXiv:hep-th/0306117]; G. Papadopoulos and D. Tsimpis, The holonomy of IIB supercovariant connection, *Classical Quantum Gravity* 20 (2003), L253–L258 [arXiv:hep-th/0307127]; A. Batrachenko, M. J. Duff, J. T. Liu and W. Y. Wen, Generalized holonomy of M-theory vacua, arXiv:hep-th/0312165; A. Batrachenko and W. Y. Wen, Generalized holonomy of supergravities with eight real supercharges, *Nuclear Phys. B* 690 (2004), 331–340 [arXiv:hep-th/0402141].
- [63] T. Z. Husain, M2-branes wrapped on holomorphic curves, *J. High Energy Phys.* 12 (2003), 037 [arXiv:hep-th/0211030]; T. Z. Husain, That’s a wrap!, *J. High Energy Phys.* 04 (2003), 053 [arXiv:hep-th/0302071].
- [64] K. Intriligator and B. Wecht, The exact superconformal R-symmetry maximizes  $a$ , *Nuclear Phys. B* 667 (2003), 183–200 [arXiv:hep-th/0304128].
- [65] P. Ivanov and S. Ivanov, SU(3)-instantons and  $G_2$ , Spin(7)-heterotic string solitons, arXiv:math.dg/0312094.
- [66] P. Kaste, R. Minasian and A. Tomasiello, Supersymmetric M-theory compactifications with fluxes on seven-manifolds and  $G$ -structures, *J. High Energy Phys.* 07 (2003), 004 [arXiv:hep-th/0303127].

- [67] P. Kaste, R. Minasian, M. Petrini and A. Tomasiello, Kaluza–Klein bundles and manifolds of exceptional holonomy, *J. High Energy Phys.* 09 (2002), 033 [arXiv:hep-th/0206213]; P. Kaste, R. Minasian, M. Petrini and A. Tomasiello, Nontrivial RR two-form field strength and SU(3)-structure, *Fortschr. Phys.* 51 (2003), 764–768 [arXiv:hep-th/0301063].
- [68] I. R. Klebanov and E. Witten, Superconformal field theory on threebranes at a Calabi–Yau singularity, *Nuclear Phys. B* 536 (1998), 199–218 [arXiv:hep-th/9807080]; D. R. Morrison and M. R. Plesser, Non-spherical horizons. I, *Adv. Theor. Math. Phys.* 3 (1999), 1–81 [arXiv:hep-th/9810201]; J. M. Figueroa-O’Farrill, Near-horizon geometries of supersymmetric branes, [arXiv:hep-th/9807149]; B. S. Acharya, J. M. Figueroa-O’Farrill, C. M. Hull and B. Spence, Branes at conical singularities and holography, *Adv. Theor. Math. Phys.* 2 (1999), 1249–1286 [arXiv:hep-th/9808014].
- [69] D. D. Joyce, *Compact manifolds with special holonomy*, Oxford University Press, Oxford 2000.
- [70] S. Kobayashi, Topology of positively pinched Kähler manifolds, *Tôhoku Math. J.* 15 (1963), 121–139.
- [71] H. Lin, O. Lunin and J. Maldacena, Bubbling AdS space and 1/2 BPS geometries, arXiv:hep-th/0409174.
- [72] A. Lukas and P. M. Saffin, M-theory compactification, fluxes and AdS<sub>4</sub>, arXiv:hep-th/0403235.
- [73] O. A. P. Mac Conamhna, Refining G-structure classifications, arXiv:hep-th/0408203.
- [74] J. M. Maldacena and C. Nunez, Supergravity description of field theories on curved manifolds and a no go theorem, *Internat. J. Modern Phys. A* 16 (2001), 822–855 [arXiv:hep-th/0007018].
- [75] P. Manousselis and G. Zoupanos, Dimensional reduction of ten-dimensional supersymmetric gauge theories in the  $\mathcal{N} = 1$ ,  $D = 4$  superfield formalism, arXiv:hep-ph/0406207.
- [76] D. Martelli and J. Sparks, G-structures, fluxes and calibrations in M-theory, *Phys. Rev. D* 68 (2003), 085014 [arXiv:hep-th/0306225].
- [77] J. Michelson, Compactifications of type IIB strings to four dimensions with non-trivial classical potential, *Nuclear Phys. B* 495 (1997), 127–148 [arXiv:hep-th/9610151].
- [78] A. Micu, Heterotic compactifications and nearly-Kähler manifolds, arXiv:hep-th/0409008.
- [79] R. Minasian and G. W. Moore, K-theory and Ramond-Ramond charge, *J. High Energy Phys.* 11 (1997), 002 [arXiv:hep-th/9710230]; E. Witten, D-branes and K-theory, *J. High Energy Phys.* 12 (1998), 019 [arXiv:hep-th/9810188]; G. W. Moore and E. Witten, Self-duality, Ramond-Ramond fields, and K-theory, *J. High Energy Phys.* 05 (2000), 032 [arXiv:hep-th/9912279].
- [80] S. B. Myers, Riemannian manifolds with positive mean curvature, *Duke Math J.* 8 (1941), 401–404.
- [81] D. N. Page and C. N. Pope, Inhomogeneous Einstein metrics on complex line bundles, *Classical Quantum Gravity* 4 (1987), 213–225.

- [82] J. Polchinski and A. Strominger, New vacua for type II string theory, *Phys. Lett. B* 388 (1996), 736–742 [arXiv:hep-th/9510227].
- [83] S. Salamon, *Riemannian geometry and holonomy groups*, Pitman Res. Notes Math. Ser. 201, Longman, London 1989.
- [84] M. B. Schulz, Superstring orientifolds with torsion: O5 orientifolds of torus fibrations and their massless spectra, arXiv:hep-th/0406001.
- [85] K. Sekigawa, On some 4-dimensional compact Einstein almost Kähler manifolds, *Math. Ann.* 271 (1985), 333–337; K. Sekigawa, On some compact Einstein almost Kähler manifolds, *J. Math. Soc. Japan* 39 (1987), 677–684.
- [86] A. Strominger, Superstrings with torsion, *Nuclear Phys. B* 274 (1986), 253–284.
- [87] T. R. Taylor and C. Vafa, RR flux on Calabi–Yau and partial supersymmetry breaking, *Phys. Lett. B* 474 (2000), 130–137 [arXiv:hep-th/9912152].
- [88] G. Tian, On Kähler–Einstein metrics on certain Kähler manifolds with  $c_1(M) > 0$ , *Invent. Math.* 89 (1987), 225–246.
- [89] G. Tian and S. T. Yau, On Kähler–Einstein metrics on complex surfaces with  $C_1 > 0$ , *Comm. Math. Phys.* 112 (1987), 175–203.
- [90] K. P. Tod, All metrics admitting super-covariantly constant spinors, *Phys. Lett. B* 121 (1983), 241–244; K. P. Tod, More on supercovariantly constant spinors, *Classical Quantum Gravity* 12 (1995), 1801–1820.



## **AdS/CFT Correspondence: Einstein Metrics and Their Conformal Boundaries**

Olivier Biquard, Editor

Since its discovery in 1997 by Maldacena, AdS/CFT correspondence has become one of the prime subjects of interest in string theory, as well as one of the main meeting points between theoretical physics and mathematics. On the physical side it provides a duality between a theory of quantum gravity and a field theory. The mathematical counterpart is the relation between Einstein metrics and their conformal boundaries. The correspondence has been intensively studied, and a lot of progress emerged from the confrontation of viewpoints between mathematics and physics.

Written by leading experts and directed at research mathematicians and theoretical physicists as well as graduate students, this volume gives an overview of this important area both in theoretical physics and in mathematics. It contains survey articles giving a broad overview of the subject and of the main questions, as well as more specialized articles providing new insight both on the Riemannian side and on the Lorentzian side of the theory.

ISBN 3-03719-013-2



9 783037 190135

[www.emis-ph.org](http://www.emis-ph.org)